Mechanised Semantic Session Typing

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VEST

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Shortcomings of Syntactic Typing

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- Must reprove **progress** and **preservation** when adding types/rules

Goal:

A “mechanisable” session type system

Solution:

A semantic session type system!

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$$\mathbf{B} \triangleq \lambda w.\, w \in \mathbb{B} \qquad\qquad \vDash b : \mathbf{B}$$

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- **Higher-Order:** Recursion, Polymorphism
- **Concurrent:** Ghost state mechanisms to reason about concurrency
- **Separation Logic:** Implicit separation of **linear** ownership
- Mechanised in **Coq** (which has **binder** support)

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Actris [Hinrichsen et al., POPL'20]

- **Dependent separation protocols (DSP):** Session type-style logical protocols
- Mechanised in **Coq**

Semantic Session Type System

- Rich extensible type system for session types
  - Term and session type equi-recursion
  - Term and session type polymorphism
  - Term and (asynchronous) session type subtyping
  - Unique and shared reference types, Copyable types, Lock types
- Full mechanisation in Coq (https://gitlab.mpi-sws.org/iris/actris)
- Supports integrating safe yet untypeable programs
- **Actris 2.0:** Subprotocols

Semantic Session Type System

Language

Language: ML-like language extended with concurrency, state and message passing

$$e \in \text{Expr} ::= v \mid x \mid \texttt{rec}\, f(x) = e \mid e_1(e_2) \mid e_1 \mid\mid e_2 \mid \texttt{ref}\ (e) \mid\ !\, e \mid e_1 \leftarrow e_2 \mid$$
$$\texttt{new\_chan}\ () \mid \texttt{send}\ e_1\ e_2 \mid \texttt{recv}\ e \mid \ldots$$

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Evaluation is performed right-to-left

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Message-passing is:

- **Binary:** Each channel have one pair of endpoints
- **Asynchronous:** `send` does not block, two buffers per endpoint pair
- **Affine:** No `close` expression, channels can be thrown away

Semantic Term Types

Types as Iris predicates:

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$$\begin{aligned} \mathsf{Type}_\bigstar &\triangleq \mathsf{Val} \to \mathsf{iProp} \\ \mathbf{Z} &\triangleq \lambda\, w.\, w \in \mathbb{Z} \\ A_1 \times A_2 &\triangleq \lambda\, w.\, \exists w_1, w_2.\, w = (w_1, w_2) * \triangleright(A_1\, w_1) * \triangleright(A_2\, w_2) \end{aligned}$$

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A \multimap B &\triangleq \lambda\, w.\, \forall v.\, \triangleright(A\, v) \mathrel{-\!\!*} \mathsf{wp}\, (w\ v)\, \{B\}
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wp $e\,\{v.\Phi\}$ dictates e does not get *stuck* and if e reduces to a value v then $\Phi\ v$ holds

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Judgement as Iris weakest precondition:

$$\Gamma \vDash e : A \dashv \Gamma' \triangleq \forall \sigma.\, (\Gamma \vDash \sigma) \mathrel{-\!\!*} \text{wp}\; e[\sigma]\,\{v. A\, v * (\Gamma' \vDash \sigma)\}$$

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Soundness: If $\varnothing \vDash e : A \dashv \Gamma$ then e does not get stuck

- Consequence of Iris's adequacy of weakest precondition

---

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Semantic Term Types - Proofs

Rules:

$$\Gamma \vDash i : \mathbf{Z}$$

$$\frac{\Gamma_2 \vDash e_1 : A_1 \dashv \Gamma_3 \qquad \Gamma_1 \vDash e_2 : A_2 \dashv \Gamma_2}{\Gamma_1 \vDash (e_1, e_2) : A_1 \times A_2 \dashv \Gamma_3}$$

If $\varnothing \vDash e : A \dashv \Gamma$
then e does not get stuck

Proofs:

```
Lemma ltyped_int Γ (i : Z) : ⊢ Γ ⊨ #i : lty_int.
Proof. iIntros "!>" (vs) "Henv /=". iApply wp_value. eauto. Qed.
```

```
Lemma ltyped_pair Γ1 Γ2 Γ3 e1 e2 A1 A2 :
  (Γ2 ⊨ e1 : A1 ⫤ Γ3) -∗ (Γ1 ⊨ e2 : A2 ⫤ Γ2) -∗
  Γ1 ⊨ (e1,e2) : A1 * A2 ⫤ Γ3.
Proof.
  iIntros "#H1 #H2". iIntros (vs) "!> HΓ /=".
  wp_apply (wp_wand with "(H2 [HΓ //])"); iIntros (w2) "[HA2 HΓ]".
  wp_apply (wp_wand with "(H1 [HΓ //])"); iIntros (w1) "[HA1 HΓ]".
  wp_pures. iFrame "HΓ". iExists w1, w2. by iFrame.
Qed.
```

```
Lemma ltyped_safety `{heapPreG Σ} e σ es σ' e' :
  (∀ `{heapG Σ}, ∃ A Γ', ⊢ ∅ ⊨ e : A ⫤ Γ') →
  rtc erased_step ([e], σ) (es, σ') → e' ∈ es →
  is_Some (to_val e') ∨ reducible e' σ'.
Proof.
  intros Hty. apply (heap_adequacy Σ NotStuck e σ (λ _, True))=> // ?.
  destruct (Hty _) as (A & Γ' & He). iIntros "_".
  iDestruct (He $!∅ with "[]") as "He"; first by rewrite /env_ltyped.
  iEval (rewrite -(subst_map_empty e)). iApply (wp_wand with "He"); auto.
Qed.
```

But what about session types?

Semantic Session Types - Definitions

Session types as a new type kind:

$$\begin{aligned} \text{Type}_\blacklozenge &\triangleq ? \\ \mathbf{!}A.\,S &\triangleq ? \\ \mathbf{?}A.\,S &\triangleq ? \\ \texttt{end} &\triangleq ? \end{aligned} \qquad \begin{aligned} \text{Type}_\bigstar &\triangleq \text{Val} \to \text{iProp} \\ \texttt{chan}\ S &\triangleq \lambda\, w.\, ? \end{aligned}$$

Requires capturing:

- **Linearity** of channel endpoint ownership
- **Delegation** of linear types / channels
- **Session fidelity** of communicated messages

Actris Dependent Separation Protocols

Session type-inspired protocols for functional correctness

	Dependent separation protocols	Syntactic session types
Example	$\mathbf{?}\,(x:\mathbb{Z})\,\langle x\rangle\{x > 10\}.\,\mathbf{?}\,\langle x+10\rangle\{\mathrm{True}\}.\,\mathbf{end}$	$\mathbf{?Z}.\,\mathbf{?Z}.\,\texttt{end}$
Usage	$c \rightarrowtail \mathit{prot}$	$c : S$

Semantic Session Types - Definitions

Session types as dependent separation protocols:

$$\begin{aligned} \text{Type}_{\blacklozenge} &\triangleq \text{iProto} \\ \mathbf{!}A.\,S &\triangleq \mathbf{!}\,(v : \text{Val})\,\langle v \rangle \{ \triangleright (A\,v) \}.\,S \\ \mathbf{?}A.\,S &\triangleq \mathbf{?}\,(v : \text{Val})\,\langle v \rangle \{ \triangleright (A\,v) \}.\,S \\ \texttt{end} &\triangleq \mathbf{end} \end{aligned}$$

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Semantic Session Types - Rules

Rules are proven as lemmas using the rules for dependent separation protocols

$$\begin{aligned} \Gamma &\vDash \texttt{new\_chan}\ () : \texttt{chan}\ S \times \texttt{chan}\ \overline{S} \dashv \Gamma \\ \Gamma, (c : \texttt{chan}\ \mathbf{!}A.\,S), (x : A) &\vDash \texttt{send}\ c\ x : \mathbf{1} \dashv \Gamma, (c : \texttt{chan}\ S) \\ \Gamma, (c : \texttt{chan}\ (\mathbf{?}A.\,S)) &\vDash \texttt{recv}\ c : A \dashv \Gamma, (c : \texttt{chan}\ S) \end{aligned}$$

Semantic Session Types - Proofs

Rule:

$$\Gamma, (c : \texttt{chan}\ (?A.\, S)) \vDash \texttt{recv}\ c : A \dashv \Gamma, (c : \texttt{chan}\ S)$$

Proof:

```
Lemma ltyped_recv Γ (x : string) A S :
  Γ !! x = Some (chan (<??> TY A; S))%lty →
  ⊢ Γ ⊨ recv x : A ⫤ <[x:=(chan S)%lty]> Γ.
Proof.
  iIntros (Hx) "!>". iIntros (vs) "HΓ"=> /=.
  iDestruct (env_ltyped_lookup _ _ _ _ (Hx) with "HΓ") as (v' Heq) "[Hc HΓ]".
  rewrite Heq.
  wp_recv (v) as "HA". iFrame "HA".
  iDestruct (env_ltyped_insert _ _ x (chan _) _ with "[Hc //] HΓ") as "HΓ"=> /=.
  by rewrite insert_delete (insert_id vs).
Qed.
```

Extensions

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Iris and **Actris** gives immediate rise to many type features

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Recursion	Guarded step-indexed recursion ($\triangleright$)
Term polymorphism	Higher-order impredicative quantifiers

Overview of features

Iris and **Actris** gives immediate rise to many type features

Linear products	Separation Conjunction ($*$)
Function types	Wand ($-\!\!*$) and weakest precondition ($\text{wp}\ e\ \{\Phi\}$)
Session types	Actris dependent separation protocols (iProto)
Unique references	Points-to connective ($\ell \mapsto v$)
Shared references	Invariants ($\boxed{P}$)
Copyable types	Persistent modality ($\square$)
Lock types	Iris's lock library
Session choice types	Actris dependent separation protocols (iProto)
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Overview of features

Iris and **Actris** gives immediate rise to many type features

Linear products	Separation Conjunction $(*)$

Subprotocols: $prot_1 \sqsubseteq prot_2$

- Generalisation of asynchronous subtyping for functional correctness
- Makes asynchronous semantics explicit by swap rule
  - $\mathbf{?}\,\langle v_1 \rangle \{P_1\}.\, \mathbf{!}\,\langle v_2 \rangle \{P_2\}.\, prot \sqsubseteq \mathbf{!}\,\langle v_2 \rangle \{P_2\}.\, \mathbf{?}\,\langle v_1 \rangle \{P_1\}.\, prot$
  - $\mathbf{?}A_1.\, \mathbf{!}A_2.\, S <: \mathbf{!}A_2.\, \mathbf{?}A_1.\, S$
- Non-trivial extension due to dependent binders and step-indexing
  - Required updates to the model of iProto

Term polymorphism	Higher-order impredicative quantifiers
Session polymorphism	Higher-order impredicative protocols binders
Term subtyping	Predicates closed under wand $(\forall v.\, A_1\ v \mathrel{-\!\!*} A_2\ v)$
Session subtyping	Actris 2.0 subprotocols $(\sqsubseteq)$

Overview of features - Definitions

Unique references: $\texttt{ref}_{\texttt{uniq}}\, A \triangleq \lambda\, w.\, \exists v.\, w \in \mathsf{Loc} * (w \mapsto v) * \triangleright(A\, v)$

Shared references: $\texttt{ref}_{\texttt{shr}}\, A \triangleq \lambda\, w.\, (w \in \mathsf{Loc}) * \boxed{\exists v.\, (w \mapsto v) * \Box(A\, v)}$

Copyable types: $\texttt{copy}\, A \triangleq \lambda\, w.\, \Box(A\, w)$

Lock types:

$$\texttt{mutex}\, A \triangleq \lambda\, w.\, \exists lk, \ell.\, (w = (lk, \ell)) * \texttt{isLock}\, lk\, (\exists v.\, (\ell \mapsto u) * \triangleright(A\, v))$$

$$\overline{\texttt{mutex}}\, A \triangleq \lambda\, w.\, \exists lk, \ell.\, (w = (lk, \ell)) * \texttt{isLock}\, lk\, (\exists v.\, (\ell \mapsto u) * \triangleright(A\, v)) * (\ell \mapsto -)$$

Session choice:

$$\oplus\{\vec{S}\} \triangleq \mathbf{!}\, (l : \mathbb{Z})\, \langle l \rangle \big\{\triangleright(l \in \mathrm{dom}(\vec{S}))\big\}.\, \vec{S}(l)$$

$$\&\{\vec{S}\} \triangleq \mathbf{?}\, (l : \mathbb{Z})\, \langle l \rangle \big\{\triangleright(l \in \mathrm{dom}(\vec{S}))\big\}.\, \vec{S}(l)$$

Recursion: $\mu\, (X : k).\, K \triangleq \mu\, (X : \mathsf{Type}_k).\, K$ (K must be contractive in X)

Polymorphism:

$$\forall (X : k).\, A \triangleq \lambda\, w.\, \forall (X : \mathsf{Type}_k).\, \mathsf{wp}\; w\, ()\, \{A\}$$

$$\exists (X : k).\, A \triangleq \lambda\, w.\, \exists (X : \mathsf{Type}_k).\, \triangleright(A\, w)$$

$$\mathbf{!}_{\vec{X}:\vec{k}}\, A.\, S \triangleq \mathbf{!}\, (\vec{X} : \vec{\mathsf{Type}_k})(v : \mathsf{Val})\, \langle v \rangle \{\triangleright(A\, v)\}.\, S$$

$$\mathbf{?}_{\vec{X}:\vec{k}}\, A.\, S \triangleq \mathbf{?}\, (\vec{X} : \vec{\mathsf{Type}_k})(v : \mathsf{Val})\, \langle v \rangle \{\triangleright(A\, v)\}.\, S$$

Term subtyping: $A <: B \triangleq \forall v.\, A\ v \mathrel{-\!\!*} B\ v$

Session subtyping: $S_1 <: S_2 \triangleq S_1 \sqsubseteq S_2$

Typing the Untypeable

An Untypeable Program

Consider the following program:

$$\vDash \lambda\, c.\, (\texttt{recv}\ c \parallel \texttt{recv}\ c) : \texttt{chan}\ (\mathbf{?Z}.\, \mathbf{?Z}.\, \texttt{end}) \multimap (\mathbf{Z} \times \mathbf{Z})$$

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Is it typeable?

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Is it typeable? No

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Is it typeable? No It violates the ownership discipline

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Is it safe?

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Is it typeable?	No	It violates the ownership discipline
Is it safe?	Yes	

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Is it safe?	Yes	Order of receives does not matter

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Really?	Well...	It could be added as an ad-hoc rule

The rule is just another lemma proven by unfolding all type-level definitions

$$(c \rightarrowtail \mathbf{?}\,(v_1 : \mathsf{Val})\,\langle v_1\rangle\{\triangleright(v_1 \in \mathbb{Z})\}.\, \mathbf{?}\,(v_2 : \mathsf{Val})\,\langle v_2\rangle\{\triangleright(v_2 \in \mathbb{Z})\}.\, \mathbf{end}) \mathrel{-\!\!*}$$
$$\mathsf{wp}\,(\texttt{recv}\ c \parallel \texttt{recv}\ c)\,\{v.\, \exists v_1, v_2.\, (v = (v_1, v_2)) * \triangleright(v_1 \in \mathbb{Z}) * \triangleright(v_2 \in \mathbb{Z})\}$$

An Untypeable Program

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$$\mathrm{wp}\ (\texttt{recv}\ c \parallel \texttt{recv}\ c)\ \{v.\, \exists v_1, v_2.\, (v = (v_1, v_2)) * \triangleright(v_1 \in \mathbb{Z}) * \triangleright(v_2 \in \mathbb{Z})\}$$

And then using Iris's ghost state machinery!

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And then using Iris's ghost state machinery! Beyond the scope of this talk

Concluding Remarks

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Semantic typing and separation logic is a good fit for mechanising session types

- **Linearity** is implicit from separation logic
- **Binders** can be inherited from underlying logic

Using a strong logic gives immediate rise to advanced features

- **Iris:** Polymorphism, recursion, locks and more
- **Actris:** Session types, session polymorphism, session subtyping

Sources:

- Paper (https://iris-project.org/pdfs/2020-actris2-submission.pdf)
- Mechanisation in Coq (https://gitlab.mpi-sws.org/iris/actris)

Questions?

Subtyping

Conventional subtyping:

$$\frac{S_1 <: S_2}{\texttt{chan}\ S_1 <: \texttt{chan}\ S_2} \qquad \frac{A_2 <: A_1 \qquad S_1 <: S_2}{\mathbf{!}A_1.\,S_1 <: \mathbf{!}A_2.\,S_2} \qquad \frac{A_1 <: A_2 \qquad S_1 <: S_2}{\mathbf{?}A_1.\,S_1 <: \mathbf{?}A_2.\,S_2}$$

Semantic Asynchronous Session Subtyping

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Asynchronous Subtyping:

$$\mathbf{?}A_1.\, \mathbf{!}A_2.\, S <: \mathbf{!}A_2.\, \mathbf{?}A_1.\, S$$

Semantic Asynchronous Session Subtyping

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Asynchronous Subtyping:

$$\mathbf{?}A_1.\, \mathbf{!}A_2.\, S <: \mathbf{!}A_2.\, \mathbf{?}A_1.\, S$$

Polymorphism subtyping:

$$\begin{aligned} \mathbf{!}_{(\vec{X}:\vec{k})}\, A.\, S &<: \mathbf{!}A[\vec{K}/\vec{X}].\, S[\vec{K}/\vec{X}] \\ \mathbf{?}A[\vec{K}/\vec{X}].\, S[\vec{K}/\vec{X}] &<: \mathbf{?}_{(\vec{X}:\vec{k})}\, A.\, S \end{aligned} \qquad \frac{S_1 <: \mathbf{!}A.\, S_2}{S_1 <: \mathbf{!}_{(\vec{X}:\vec{k})} A.\, S_2} \qquad \frac{\mathbf{?}A.\, S_1 <: S_2}{\mathbf{?}_{(\vec{X}:\vec{k})} A.\, S_1 <: S_2}$$

Semantic Asynchronous Session Subtyping - Example

Goal:

$$\mu\,(\mathit{rec} : \blacklozenge).\,!_{(X,Y:\bigstar)}\,(X \multimap Y).\,!X.\,?Y.\,\mathit{rec} <: \mu\,(\mathit{rec} : \blacklozenge).\,!_{(X_1,X_2:\bigstar)}\,(X_1 \multimap \mathbf{B}).\,!X_1.\,!(X_2 \multimap \mathbf{Z}).\,!X_2.\,?\mathbf{B}.\,?\mathbf{Z}.\,\mathit{rec}$$

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Derivation:

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Derivation:

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Rules:

S-ELIM

$$\frac{S_1 <: \mathbf{!}A.\,S_2}{S_1 <: \mathbf{!}_{(\vec{X}:\vec{k})}A.\,S_2}$$

S-INTRO

$$\mathbf{!}_{(\vec{X}:\vec{k})}\,A.\,S <: \mathbf{!}A[\vec{K}/\vec{X}].\,S[\vec{K}/\vec{X}]$$

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\end{aligned}$$

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