Solutions of the exercises on
Sets, Relations and Functions

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Exercises on slide 11

Exercise 1
Argue that $A$ and $\bar{A}$ are disjoint.

Solution
By definition of the complement, $\bar{A}$ is a set of those element from the universal set $\mathcal{U}$, which are not in $A$, so if $x \in A$ then $x \notin \bar{A}$ and if $x \notin A$ then $x \not\in A$, thus there is no such $x$ that $x \in A$ and $x \in \bar{A}$, therefore $A \cap \bar{A} = \emptyset$.

Exercise 2
Let $\mathcal{U} = \mathbb{N}$. What is the complement of $\{x : x^2 - 3x - 4 = 0\}$? What if $\mathcal{U} = \mathbb{Q}$?

Solution
$A = \{x : x^2 - 3x - 4 = 0\} = \{x : (x - 4)(x + 1) = 0\}$, then $\bar{A} = \{x : x^2 - 3x - 4 \neq 0\}$. Thus for $\mathcal{U} = \mathbb{N}$, $A = \{4\}$ and $\bar{A} = \{0, 1, 2, 3, 5, 6, 7, \ldots\}$. And for $\mathcal{U} = \mathbb{Q}$, $A = \{-1, 4\}$ and $\bar{A} = \mathbb{Q} \setminus \{-1, 4\}$.

Exercise on slide 12
Give an intuitive explanation of $\mathcal{P}(\mathbb{N})$.

Solution
$\mathcal{P}(\mathbb{N})$ is a set of all subset of $\mathbb{N}$ including the empty set $\emptyset$ and $\mathbb{N}$ itself.
$\mathcal{P}(\mathbb{N}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \ldots, \{0, 1\}, \{0, 2\}, \ldots\{0, 1, 2\}, \ldots\{0, 1, 2, \ldots, n, \ldots\}, \ldots, \mathbb{N}\}$. 
Exercises on slide 13

Exercise 1
Give a partition of the real numbers \( \mathbb{R} \).

Solution
An example of a partition of \( \mathbb{R} \) can be \( \{A, B, C\} \), where \( A = \{x : x > 0\} \), \( B = \{0\} \) and \( C = \{x : x < 0\} \). It is because \( A \), \( B \) and \( C \) are not empty sets, they are pairwise disjoint and their union is equal to \( \mathbb{R} \).

Exercise 2
Do there exist a partition of \( \mathbb{N} \)?

Solution
The partition of \( \mathbb{N} \) is \( \{\} \).
Let \( \exists A_i \), that \( \{A_i : i \in I\} \) is a partition of \( \mathbb{N} \). Then by the definition of a partition:
(a) \( A_i \neq \emptyset , i \in I \)
(b) \( \bigcup_{i \in I} A_i = \emptyset \)
(c) \( A_i \cap A_j = \emptyset , i \neq j , i , j \in I \)
As it follows from (b) \( A_i = \emptyset \) for all \( i \in I \), that contradicts to (a). Thus our assumption about existence of \( A_i \) was not true. Hence there are not such \( A_i \) that \( \{A_i : i \in I\} \) is a partition of \( \emptyset \), therefore the only partition of \( \emptyset \) is \( \emptyset \).

Exercise on slide 14
Give an example where \( (a, b) \in A \) but \( (b, a) \notin A \).

Solution
Let \( A = \{(x, y) : x \text{ is a father of } y\} \). Then if Adam is a father of Bob, \((Adam, Bob) \in A\) but \((Bob, Adam) \notin A\), because Bob is a son of Adam, and so he cannot be his father.

Exercise on slide 20
Compute \((R_3 \circ R_2) \circ R_1\) (lecture slide 20).

Solution
By the theorem on the lecture slide 20 \( R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1 \).
Thus \( (R_3 \circ R_2) \circ R_1 = \{(Adam, 30), (Bob, 63), (Chris, 52), (Dave, 30), (Eve, 63)\} \)

Exercise on slide 21
Why is not \(<\) on \( \mathbb{N} \) an equivalence relation? Why is not \(\leq\)?
Solution

$<$ is not an equivalence relation on $\mathbb{N}$, because it is not reflexive. It follows from the fact that $\forall a \in \mathbb{N} \ a \not< a$.

$\leq$ is not an equivalence relation on $\mathbb{N}$, because it is not symmetric. It follows from the fact that if $a \leq b$ and $b \leq a$ then $a = b$, for all $a, b \in \mathbb{N}$.

Exercise on slide 24

Draw a tree which shows the $\prec$-ordering of the elements in $R_3 \circ (R_2 \circ R_1)$.

Solution

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(Adam,30)
   /   \
  /     \n(Bob,63) (Chris,52) (Dave,30)
  |     |     |
(Eve,63)
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Exercise on slide 26

Why is $\subseteq$ not a total order on $\mathcal{P}(A)$ if $A$ contains at least two elements?

Solution

It is because not any two elements in $\mathcal{P}(A)$ can be related. For example, if $A = \{a, b\}$, then $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{a, b\}$. Here two sets $\{a\}$ and $\{b\}$ cannot be related by $\subseteq$ order.

Exercise on slide 29

Let $f(x) = 2x + 3$ and $g(x) = 3x + 2$ be functions on $\mathbb{N}$. What is $(g \circ f)(x)$?

Solution

$$(g \circ f)(x) = 3(f(x)) + 2 = 3(2x + 3) + 2 = 6x + 11.$$ 

Exercises on slide 34

Exercise 2

Argue that for all $n \in \mathbb{Z}^+$ the relation $R_n$ on $\mathbb{Z}^+$, defined by $aR_nb$ if and only if $a \% n = b \% n$, is an equivalence relation.
Solution
In order to be an equivalence relation $R_n$ must be reflexive(i), symmetric(ii) and transitive(iii).
(i) $\forall a \in \mathbb{Z}^+: a \% n = a \% n$. Therefore $R_n$ is reflexive.
(ii) $\forall a, b \in \mathbb{Z}^+: a \% n = b \% n$ implies that $b \% n = a \% n$. Therefore $R_n$ is symmetric.
(iii) $\forall a, b, c \in \mathbb{Z}^+: a \% n = b \% n$ and $b \% n = c \% n$ implies that $a \% n = c \% n$. Therefore $R_n$ is transitive.

Exercise 3

Argue why an equivalence relation that is also a function must be the identity.

Solution

Let $R$ be an equivalence relation on $A$ and a function $R : A \to A$. Then by the definition of a function, for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, which means that $aRb$. As it follows from the fact that an equivalence relation is reflexive, for every $a \in A$ $aRa$. Hence for every $a \in A$, there is one and only one $b \in A$ so that $(a, b) \in R$, and such $b = a$. Thus $R$ is the identity.

Exercise 3.2.3 on page 80 in DM for NT

Let $\mathcal{U} = \{x : x \text{ is an integer and } 2 \leq x \leq 10\}$. In each of the following cases, determine whether $A \subseteq B, B \subseteq A$, both or neither:
(i) $A = \{x : x \text{ is odd}\} B = \{x : x \text{ is a multiple of 3}\}$
(ii) $A = \{x : x \text{ is even}\} B = \{x : x^2 \text{ is even}\}$
(iii) $A = \{x : x \text{ is even}\} B = \{x : x \text{ is a power of 2}\}$
(iv) $A = \{x : 2x + 1 > 7\} B = \{x : x^2 > 20\}$
(v) $A = \{x : \sqrt{x} \in \mathbb{Z}\} B = \{x : x \text{ is a power of 2 or 3}\}$
(vi) $A = \{x : \sqrt{x} \leq 2\} B = \{x : x \text{ is a perfect square}\}$
(vii) $A = \{x : x^2 - 3x + 2 = 0\} B = \{x : x + 7 \text{ is a perfect square}\}$.

Solution

$\mathcal{U} = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(i) neither, $A = \{3, 5, 7, 9\} B = \{3, 6, 9\}$
(ii) both, $A = \{2, 4, 6, 8, 10\} B = \{2, 4, 6, 8, 10\}$
(iii) $B \subseteq A$, $A = \{2, 4, 6, 8, 10\} B = \{2, 4, 8\}$
(iv) $B \subseteq A$, $A = \{4, 5, 6, 7, 8, 9, 10\} B = \{5, 6, 7, 8, 9, 10\}$
(v) $A \subseteq B$, $A = \{4, 9\} B = \{2, 3, 4, 8, 9\}$
(vi) neither, $A = \{2, 3, 4\} B = \{4, 9\}$
(vii) both, $A = \{2\} B = \{2\}$. 
Exercise 3.2.8 on page 81 in DM for NT

(i) Prove that, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(ii) Deduce that, if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$, then $A = B = C$.

Solution

(i) Let $x \in A$, then it follows from $A \subseteq B$ that $x \in B$, then it follows from $B \subseteq C$ that $x \in C$. This proves that every element of $A$ also belongs to $C$, so $A \subseteq C$.
(ii) $A \subseteq B$, $B \subseteq C$, so it follows from (i) that $A \subseteq C$. If $A \subseteq C$ and $C \subseteq A$ then by the theorem on the lecture slide $8 A = C$. From $A = C$, $A \subseteq B$ and $B \subseteq C$ follows that $C \subseteq B$ and $B \subseteq C$, and thus by the same theorem $B = C$. Therefore $A = B = C$.

Exercise 3.2.10 on page 81 in DM for NT

Consider the set $R$ of all sets which are not elements of themselves. That is, $R = \{A : A$ is a set and $A \notin A\}$.
Find a set which is an element of $R$. Can you find a set which is not an element of $R$?
Explain why $R$ is not a well defined set. (Hint: is $R$ itself an element of $R$?)

Solution

Let $B$ is a set and $B \in R$, then by definition of $R B \notin B$. An example of such set $B$ can be $B = \{1\}$ and many others, because usually $B \notin B$, like $\{1\} \notin \{1\}$.
Let us now find a set $C$ which is not an element of $R$, so $C \in C$ must hold. An example of such set can be $C = \{\cdots \{ \{1\} \} \cdots \}$, because $\{\cdots \{ \{1\} \} \cdots \} \in \{\cdots \{ \{1\} \} \cdots \}$.
$R$ is not well defined, because assuming that $R \in R$, it follows from the definition of $R$ that $R \notin R$, and assuming that $R \notin R$, it follows that $R \in R$. Such definition of $R$ leads to a contradiction.

Exercise 3.5.4 on page 106 in DM for NT

Which of the following are partitions of $\mathbb{R}$, the set of real numbers? Explain your answers.
(i) $\{I_n : n \in \mathbb{Z}\}$, where $I_n = \{x \in \mathbb{R} : n \leq x \leq n + 1\}$.
(ii) $\{J_n : n \in \mathbb{Z}\}$, where $J_n = \{x \in \mathbb{R} : n \leq x < n + 1\}$.
(iii) $\{K_n : n \in \mathbb{Z}\}$, where $K_n = \{x \in \mathbb{R} : n < x < n + 1\}$.

Solution

(i) $\{I_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $I_n \cap I_{n+1} \neq \emptyset$, and it follows from the fact that $\exists x = n + 1 : x \in I_n$ and $x \in I_{n+1}$.

(ii) $\{J_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$, because $\forall n \in \mathbb{Z}, J_n \neq \emptyset$, $\bigcup_{n \in \mathbb{Z}} J_n = \mathbb{R}$ and $J_i \cap J_j = \emptyset$ if $i \neq j$ for all $i, j \in \mathbb{Z}$.

(iii) $\{K_n : n \in \mathbb{Z}\}$ is not a partition of $\mathbb{R}$, because $\bigcup_{n \in \mathbb{Z}} K_n \neq \mathbb{R}$, and it follows from the fact that $\forall i \in \mathbb{Z}, i \in \mathbb{R}$ and $i \notin K_n \forall n \in \mathbb{Z}$.
Exercise 5 on page 140 in DM with Combinatorics

Which of the following functions, whose domain and codomain are the real line, are one-to-one, which are onto, and which have inverses:

(a) \(f(x) = |x|\)
(b) \(f(x) = x^2 + 4\)
(c) \(f(x) = x^3 + 6\)
(d) \(f(x) = x + |x|\)
(e) \(f(x) = x(x - 2)(x + 2)\)

Solution

(a) \(f(x) = |x|\).
This function is not one-to-one, because \(\exists x_1 \text{ and } \exists x_2 : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2\), for example \(x_1 = 3\) and \(x_2 = -3\).
This function is not onto, because there exists such \(y\), that for every \(x : f(x) \neq y\), for example, \(y < 0\), where \(f(x) \geq 0\) for every \(x\).
This function can have inverse \(f^{-1}(y)\) only on \(y \in [0, +\infty)\), because \(f(x) \geq 0\) for all \(x \in (-\infty, +\infty)\), moreover \(f^{-1}(y) = \pm y\), that is not a function.

(b) \(f(x) = x^2 + 4\)
This function is not one-to-one, because \(\exists x_1 \text{ and } \exists x_2 : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2\), for example \(x_1 = 3\) and \(x_2 = -3\).
This function is not onto, because there exists such \(y\), that for every \(x : f(x) \neq y\), for example, \(y < 4\), where \(f(x) \geq 4\) for every \(x\).
This function can have inverse \(f^{-1}(y)\) only on \(y \in [4, +\infty)\), because \(f(x) \geq 4\) for all \(x \in (-\infty, +\infty)\), moreover \(f^{-1}(y) = \pm \sqrt{y-4}\), that is not a function.

(c) \(f(x) = x^3 + 6\)
This function is one-to-one, because \(\forall x_1 \text{ and } \forall x_2 : f(x_1) = f(x_2)\) implies that \(x_1 = x_2\), as it follows from \(x_1^3 + 6 = x_2^3 + 6\) that \(x_1 = x_2\).
This function is onto, because for every \(y\), there exists \(x : f(x) = y\).
This function have inverse \(f^{-1}(y)\) on \(y \in (-\infty, +\infty)\), and \(f^{-1}(y) = \sqrt[3]{y-6}\), that is a function.

(d) \(f(x) = x + |x|\)
This function is not one-to-one, because \(\exists x_1 \text{ and } \exists x_2 : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2\), for example \(x_1 = -3\) and \(x_2 = -4\).
This function is not onto, because there exists such \(y\), that for every \(x : f(x) \neq y\), for example, \(y < 0\), where \(f(x) \geq 0\) for every \(x\).
This function can have inverse \(f^{-1}(y)\) only on \(y \in [0, +\infty)\), because \(f(x) \geq 0\) for all \(x \in (-\infty, +\infty)\), moreover for \(y > 0\) the inverse is defined as \(f^{-1}(y) = \frac{y}{2}\), and for \(y = 0\) \(f^{-1}(y) = a\), where \(a\) can be any real number that \(\leq 0\), so such inverse is not a function.

(e) \(f(x) = x(x - 2)(x + 2)\)
This function is not one-to-one, because \(\exists x_1 \text{ and } \exists x_2 : f(x_1) = f(x_2) \text{ and } x_1 \neq x_2\), for example \(x_1 = 0\) and \(x_2 = 2\).
This function is onto, because for every \(y\), there exists \(x : f(x) = y\).
This function have inverse \(f^{-1}(y)\) on \(y \in (-\infty, +\infty)\), and \(f^{-1}(0) = 0, 2\) or \(-2\), that is not a
Exercise 5 on page 160 in DM with Combinatorics

Show that the set $A = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countably infinite.

Solution

This set $A$ is countably infinite, because there exists a bijection $f : A \rightarrow \mathbb{Z}^+$, where $f(a) = a + 11$ for $a \in A$. 