Foundations of Computing

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Instructions

What to bring. You can bring any written aid you want. This includes the course book and a dictionary. In fact, these two things are the only aids that make sense, so we recommend you bring them and only them. But if you want to bring other books, notes, print-out of code, old exams, or today's newspaper you can do so. (It won't help.)

You cannot bring electronic aids (such as a laptop) or communication devices (such as a mobile phone). If you really want, you can bring an old-fashioned pocket calculator (not one that solves recurrence relations).

What do check. In the multiple-choice questions, there is one and only one correct answer. However, to demonstrate partial knowledge, you are allowed to check 2 or more boxes, but this earns you less than full points for that question. If the correct answer is not among your checked boxes, the score will even be negative. If you do not check anything (or check *all* boxes) your score is 0. Also, if you check boxes at random, your expected score is 0. For more details, read [Gudmund Skovbjerg Frandsen, Michael I. Schwartzbach: A singular choice for multiple choice. SIGCSE Bulletin 38(4): 34–38 (2006)].

1 Algorithms and data structures only

1.

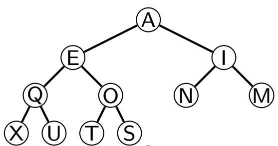
(a) (*1 pt.*) Consider a queue, and the below sequence of operations. A letter x means "insert x" and "$*$" means "get".

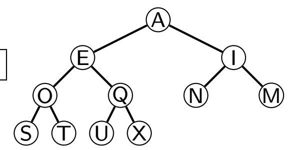
E * X A * * * M Q * U E S * T * I O N *

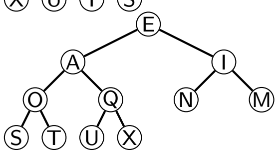
Give the sequence of values returned by the get operations.

- [ ] E X A M Q U E S T I O N null
- [ ] E X A null M Q U E S T I O N
- [x] E X A null M Q U E
- [ ] E X A M null Q U E
- [ ] E X A M null Q U E S null

(b) (*1 pt.*) Which of the following is *not* a binary min-heap for the character set E X A M Q U S T I O N (where A is smallest and Z is largest)?

- [ ] 

- [ ] 

- [x] 

(c) (*1 pt.*) Which of these data structures supports binary search?

- [ ] heap
- [ ] stack
- [x] sorted array
- [ ] queue
- [ ] unbounded array

(d) (*1 pt.*) Extending by 1 the size of an unbounded array (Java ArrayList) takes, in the worst case, time:

- [ ] $O(1)$
- [ ] $O(\log n)$
- [x] $O(n)$
- [ ] $O(n^2)$

(e) (*1 pt.*) Which of the following data structures can *not* be a cuckoo hash table for the set {a,c,e,g,i,k,m,o,q}?

- [ ]

c	a	e	g	i	k	m	o
				q			

- [ ]

a	c	e	g	i	k	m	o
				q			

- [x]

a	c	e	g	i	k	m	o
e				q		a	

- [ ]

				q			
a	c	e	g	i	k	m	o

(f) (*1 pt.*) Consider the following procedure:

int *fibb*(int x) {
    if $(x < 3)$ **return** 1;
    else return *fibb*$(x-2)$+*fibb*$(x-3)$;
}

After a call fibb(7), how many times is fibb called recursively?

- [x] 8

☐ 10
☐ 11
☐ 13

(g) (*1 pt.*) Suppose quicksort (with randomized pivot selection) is run on an array of length 9 containing the characters S O R T M E N O W. Which of the following arrays may be the contents of the array after the first partitioning step?

☐ | O | R | S | T | E | M | N | O | W |

☐ | E | M | N | O | O | R | S | T | W |

☑ | O | O | N | E | M | R | T | S | W |

☐ | E | S | O | R | T | M | N | O | W |

(h) (*1 pt.*) Which of the following statements about heapsort is true?

☑ Is in-place
☐ Requires recursive calls
☐ Is cache-friendly
☐ Requires a random number generator

(i) (*1 pt.*) What is the asymptotic running time of the following pseudocode:

for $i = 1$ to n **do**
    for $j = 1$ to n **do**
        for $k = 1$ to n **do**
            if $(i < j)$ **and** $(j < k)$ **and** $A[i] < A[j]$ **and** $A[j] < A[k]$ **then** $c = c + 1$;

☐ $O(\log n)$
☐ $O(n \log n)$
☐ $O(n^2)$
☑ $O(n^3)$

(j) (*1 pt.*) Compared to the pseudocode in the previous question, the below pseudocode is:

for $i = 1$ to n **do**
    for $j = i + 1$ to n **do**
        for $k = j + 1$ to n **do**
            if $A[i] < A[j]$ **and** $A[j] < A[k]$ **then** $c = c + 1$;

☐ Asymptotically faster
☑ Asymptotically the same speed
☐ Asymptotically slower

(k) (*1 pt.*) Increasing the input size of an algorithm running in time $\Theta(n^3)$ by a factor 2, is expected to increase the running time by a factor:

☐ 2
☐ 3
☐ 6
☑ 8

(l) (*1 pt.*) In this question we use % to denote the modulo operation, and / to denote integer division (i.e., these operators work like in Java). The variable a is known to be a positive integer.

```
x = a
s = ""
while x > 0 do
    if (x%2 == 1) then s = "1" + s;
    else s = "0" + s;
    x = x/2
endwhile
```

Which of the following invariants holds for the while loop?

- [ ] s is the least significant bits in the binary representation of x
- [x] s is the least significant bits in the binary representation of a
- [ ] s is the most significant bits in the binary representation of x
- [ ] s is the most significant bits in the binary representation of a

(m) (*1 pt.*) Consider this procedure which works on a binary tree (implemented as a class `Treenode`, with variables `left` and `right` denoting the children, and using `null` to indicate lack of a child):

```
int depth(Treenode n) {
    if (n == null) return 0;
    else return <expression>;
}
```

What should be substituted for <expression> such that $depth(r)$ computes the depth of a binary tree with root node r?

- [ ] 1+depth(n.left)+depth(n.right)
- [ ] 1+min(depth(n.left),depth(n.right))
- [x] 1+max(depth(n.right),depth(n.left))
- [ ] 1+max(depth(n.left)-depth(n.right),0)

(n) (*1 pt.*) Given two programs that use the same number of computation steps, the fastest will be the one that:

- [ ] Uses an amortized data structure
- [x] Makes fewer random memory accesses
- [ ] Makes use of external memory
- [ ] Is based on sorting

(o) (*1 pt.*) Which data structure for a set S should be used to get the fastest implementation of the following code, assuming S is initially empty:

```
for i = 1 to n
    if search(A[i], S)==A[i] then remove(A[i], S)
    else insert(A[i], S)
```

- [ ] Linked list
- [ ] Search tree
- [x] Hash set
- [ ] Unbounded array (sorted)

(p) (*1 pt.*) On a separate piece of paper, draw the trie for the following 5 binary strings:

```
100101
111
101101
```

```
001101
1010
```

2. The new metro of Grid City always has exactly 1, 2, or 3 minutes between two consecutive stops. Also, it always takes exactly 1 minute to change from one train to another. You are given the task of designing an efficient algorithm for finding the fastest route in the metro system, which has n stations and m connections between adjacent stations. Use a separate piece of paper to answer the following:

(a) (*3 pt.*) Argue that the fastest path problem can be modeled using an unweighted graph. Give an example of a small graph with weights, and the corresponding unweighted graph.

(b) (*2 pt.*) You expect that most of the searches will be on travel to or from one of the major metro stations. Suppose there are k such stations, and n stations in total. Describe how to construct a data structure that can report the shortest path to and from any of these k stations in $O(1)$ time. You may assume that the metro system does not change.

(c) (*1 pt.*) For the data structure in the previous part, what is the space usage (in terms of n, m, and k), and what is the time required to construct it?

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2 Discrete Mathematics only

1. Answer the following multiple choice questions:

(a) (*1 pt.*) Which of the following propositions is a *tautology*:

- [x] $p \to (q \lor p)$
- [ ] $(p \lor q) \to q$
- [ ] $p \to q$
- [ ] $p \to (q \land p)$

(b) (*1 pt.*) Which of the following relations is *not* a *function*:

- [ ] $R_1 = \{(1,2),(2,3),(3,5),(4,6)\}$ where $R_1 \subseteq \mathbb{N} \times \mathbb{N}$
- [x] $R_2 = \{(a,ab),(b,aa),(a,a)\}$ where $R_2 \subseteq \{a,b\}^*$
- [ ] $R_3 = \{(1,a),(5,a)\}$ where $R_3 \subseteq \mathbb{N} \times \{a,b\}^*$
- [ ] $R_4 = \{(a,ab)\}$ where $R_4 \subseteq \{a,b\}^* \times \{a,b\}^*$

(c) (*1 pt.*) Which of the following functions from $\mathbb{Z}$ to $\mathbb{Z}$ is a *bijection*:

- [ ] $f_1(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$
- [x] $f_2(n) = n$
- [ ] $f_3(n) = n^2$
- [ ] $f_4(n) = \begin{cases} 0 & \text{if } n = 1 \\ 1 & \text{if } n = 0 \\ n+1 & \text{otherwise} \end{cases}$

(d) (*1 pt.*) Given the sets $A = \{1,2,3,4,5\}$ and $B = \{a,b,c\}$, what is the number of *onto* functions from A to B such that

$$(i)\ f(2) = f(3) = a \qquad (ii)\ f(x) \neq a \text{ if } x \notin \{2,3\}$$

?

- [x] 6
- [ ] $\binom{5}{3}$
- [ ] 8
- [ ] 3^5

(e) (*1 pt.*) Father Christmas comes to a party with 3 different children. He carries 4 different presents and 9 different Christmas cards. In how many different combination can he distribute all presents and cards to the 3 children assuming that each child gets exactly 1 card and exactly 1 present?

- [x] $4 \cdot 3 \cdot 2 \cdot 9 \cdot 8 \cdot 7$
- [ ] $4^3 + \binom{9}{3}$
- [ ] $\binom{4}{3} \cdot \binom{9}{3}$
- [ ] $4^3 + (9 \cdot 8 \cdot 7)$

(f) (*1 pt.*) Which of the following functions can be shown to be $O(x^2)$ given witnesses $C = 2$ and $k = 4$?

- ☐ $x^2 + 891$
- ☐ $(x-1)(x+1) + 32$
- ☐ $(x-1)^4 \cdot \sqrt{x} + 15$
- ☑ $x + 16$

(g) (*1 pt.*) Which of the following is the characteristic equation of the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}$$

?

- ☑ $r^2 - 4r + 4 = 0$
- ☐ $r_1 = r_2 = 2$
- ☐ $r^2 - 4r - 4 = 0$
- ☐ It doesn't exist

(h) (*1 pt.*) Let X the number of heads obtained after independently tossing a coin 10 times such that $P(H) = P(T) = \frac{1}{2}$. Then:

- ☐ $P(X = 1) < P(X = 9)$
- ☑ $P(X = 1) = P(X = 9)$
- ☐ $P(X = 1) > P(X = 9)$
- ☐ $P(X = 1) > P(X = 9) - P(X = 10)$

(i) (*1 pt.*) Suppose that 4 green balls, 2 red balls, 8 black balls and 16 pink balls are placed in a box and a ball is randomly extracted. Moreover, let E be the event "the extracted ball is red or black", F be the event "the extracted ball is pink or green" and G be the event "the extracted ball is red or green". Which of the following events are *independent*:

- ☐ E and F
- ☑ E and G
- ☐ F and G
- ☐ no events can be independent

(j) (*1 pt.*) In year 2052, management has decided to award an iPhone 50G-Teleport to the best teacher and the best student at ITU. In order to choose the winners, all students must vote for two teachers and all teachers must express a single vote for their favourite student. Given the set P of people at ITU (consisting of both students and teachers), the relation $V \subseteq P \times P$ expresses whether some person p_1 has voted for some other person p_2. Moreover, the relation M relates two persons whenever they have voted for each other. Which of the following statements is true?

- ☐ V is an equivalence relation
- ☐ V is reflexive
- ☑ M is symmetric
- ☐ V and M are functions

(k) (*1 pt.*) Consider the following boolean expression:

$$\overline{z} + xyz + x\,\overline{y}\,\overline{z} + x\overline{y}z$$

Which of the following expressions is it equivalent to?

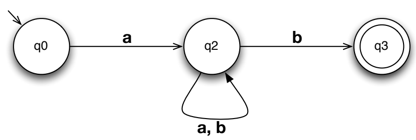
- [ ] $\overline{z} + xy + x\,\overline{y}\,\overline{z}$
- [ ] $\overline{z} + xyz$
- [x] $\overline{z} + xz$
- [ ] 1

(l) (*1 pt.*) Consider a grammar G with alphabet $\{a, b\}$, non-terminals $\{S, A\}$, starting symbol S and productions given as:

$$S \rightarrow aA \qquad\qquad A \rightarrow abS \quad | \quad b$$

Which of the following automata accepts the language generated by G?

- [ ]



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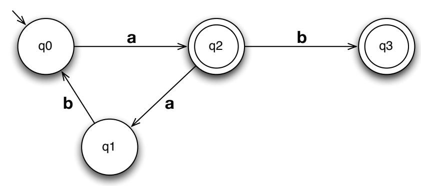
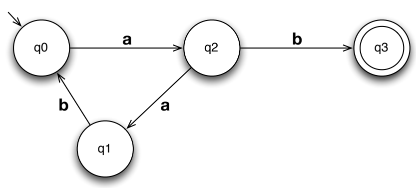


- [x]



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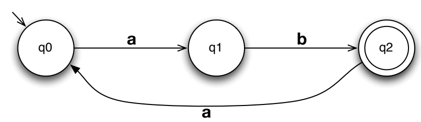

(m) (*1 pt.*) Consider the following program C where a is an array of length n containing some pre-assigned integers:

```
i=0;  s=0;
while  i<n  {
   s=s+a[i];
   i=i+1;
}
```

Which of the following statements is true?

- [ ] The loop invariant is $a = i + 1$
- [x] The loop invariant is $s = \sum_{j=0}^{i} a[i]$
- [ ] $\{i = 0 \wedge s = 0\} \quad C \quad \{s = a[i]^n\}$
- [ ] The program never terminates

2. Answer the following question on a separate piece of paper. Be brief but precise, your correct use of mathematical notation and formalism is an important aspect of this question.

(a) (*4 pt.*) Prove by mathematical induction that for any positive integer $n \in \{1, 2, 3, 4, 5, \ldots\}$ we have that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Hint. Remember that $n^2 + 2n + 1 = (n+1)^2$.