Relations as Least-Fixed Points of Inference Systems

Relations (e.g., $\vdash_{even}$, '≤', $\vdash_{EXP}$) are often *defined* as the ***least fixed point*** of functions *derived* from inference systems. :-) I'll explain, using the "even" relation as an example…

An Inference System:
An *inference system* is used to specify a relation and it consists of axiom(s) and rule(s). An *axiom* specifies a[n unconditional] member of the relation and is written like this:

$$\overline{\vdash_{even}\ 2}$$

(with a bar over the assertion) meaning that two *is* even - i.e., that 2 is in the even-relation (formally that $2 \in$ '$\vdash_{even}$'). A *rule* specifies a conditional member of the relation and is usually written like this:

$$\frac{\vdash_{even}\ n}{\vdash_{even}\ m}\quad m = n+2$$

(with a number of "assumption(s)" on top of the bar, *one* "conclusion" underneath the bar, and possibly with "side-condition(s)" on the right of the bar) meaning that *"IF n is in the even relation, AND that m equals n plus 2, THEN m is in the even relation"*.

An Inference System *specifies* a Relation and *induces* a Function:
The axiom and rule together form the inference system which *specifies* the '$\vdash_{even}$' relation (with signature '$\vdash_{even}$' $\subseteq \mathcal{P}(N)$). The inference system *induces* a function, $F : \mathcal{P}(N) \to \mathcal{P}(N)$ which takes a relation (as argument) and gives a relation (as result):

$$F(S) = \{2\} \cup \{\, m \mid n \in S,\ m=n+2\}$$

The relation '$\vdash_{even}$' is really ***defined as*** the least *fixed point* of this function (which we know how to compute)! :-)

Start with the bottom element (in the $\mathcal{P}(N)$ lattice - which is $\emptyset$) and compute $F^i(\emptyset)$ until we hit the unique least fixed point (which we are guaranteed to hit since $\mathcal{P}(N)$ is a lattice and F is monotone):

$$\emptyset \subseteq F(\emptyset) = \{2\} \subseteq F(\{2\}) = \{2,4\} \subseteq F(\{2,4\}) = \{2,4,6\} \subseteq F(\{2,4,6\}) = \{2,4,6,8\} \subseteq \ldots$$

Here, the lattice does not have finite height, so the computation is not guaranteed to converge in finite time, but the least fixed point result is the set including all the even numbers.

Of course, this is not the only fixed point of "F". The set {0, 1, 2, 3, 4, 5, 6, …} including *all* natural numbers N is also a (bigger) fixed point of "F".

Another Example '≤' ('≤' $\subseteq \mathcal{P}(\mathrm{N}) \times \mathcal{P}(\mathrm{N})$):

This relation can be specified by the following inference system (with one axiom and two rules):

axiom 1:	*rule 1:*	*rule 2:*
______ 0 ≤ 0	n ≤ m ------ m′ = m+1 n ≤ m′	n ≤ m ------- n′ = n+1, m′ = m+1 n′ ≤ m′

It induces the following function F : $\mathcal{P}(\mathrm{N}) \times \mathcal{P}(\mathrm{N}) \to \mathcal{P}(\mathrm{N}) \times \mathcal{P}(\mathrm{N})$:

$F(R) = \{ (0,0) \} \cup \{ (n,m') \mid (n,m) \in R, m'=m+1 \} \cup \{ (n',m') \mid (n,m) \in R, n'=n+1, m'=m+1 \}$

And we can compute the least fixed point of 'F':

$$\emptyset \subseteq F(\emptyset) = \{ (0,0) \} \subseteq F(\{(0,0)\}) = \{ (0,0), (0,1), (1,1) \} \subseteq \ldots$$

Intuitively, step "i" includes anything that can be established using "i" number of axioms and rules.

Yet Another Example '$\vdash_{EXP}$' ('$\vdash_{EXP}$' $\subseteq \mathcal{P}(\mathrm{ASCII}^*)$):

The BNF specification:

$$E ::= \mathtt{n} \mid \mathtt{v} \mid E + E$$

is really an inference system and specifies a relation (a set), we'll call *EXP* which we may use to determine whether an element, x, is an expression (i.e., if $x \in EXP$):

The inference system is comprised of two axioms and one rule:

axiom 1:	*axiom 2:*	*rule 1:*
____ |- n	____ |- v	$\vdash E_1$, $\vdash E_2$ -------------- $\vdash E_1 + E_2$

…and induces a function: F : $\mathcal{P}(\mathrm{ASCII}^*) \to \mathcal{P}(\mathrm{ASCII}^*)$:

$$F(X) = \{\mathtt{n}\} \cup \{\mathtt{v}\} \cup \{ E_1+E_2 \mid E_1 \in X, E_2 \in X \}$$

…and we obtain the least fixed point *EXP* by computing:

$$\emptyset \subseteq F(\emptyset) = \{ \mathtt{n}, \mathtt{v} \} \subseteq F(\{ \mathtt{n}, \mathtt{v} \}) = \{ \mathtt{n}, \mathtt{v}, \mathtt{n+n}, \mathtt{n+v}, \mathtt{v+n}, \mathtt{v+v} \} \subseteq \ldots$$

…which converges on the set of all expressions of arbitrary size. :-)