Classifying k-Edge Colouring for H-free Graphs*

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Abstract. A graph is H-free if it does not contain an induced subgraph isomorphic to H. For every integer k and every graph H, we determine the computational complexity of k-Edge Colouring for H-free graphs.

1 Introduction

A graph $G = (V, E)$ is *k-edge colourable* for some integer k if there exists a mapping $c : E \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ for any two edges e and f of G that have a common end-vertex. The *chromatic index* of G is the smallest integer k such that G is k-edge colourable. Vizing proved the following classical result.

Theorem 1 ([27]). *The chromatic index of a graph G with maximum degree Δ is either Δ or $\Delta + 1$.*

The Edge Colouring problem is to decide if a given graph G is k-edge colourable for some given integer k. Note that (G, k) is a yes-instance if G has maximum degree at most $k - 1$ by Theorem 1 and that (G, k) is a no-instance if G has maximum degree at least $k + 1$. If k is *fixed*, that is, k is not part of the input, then we denote the problem by k-Edge Colouring. It is trivial to solve this problem for $k = 2$. However, the problem is NP-complete if $k \geq 3$, as shown by Holyer for $k = 3$ and by Leven and Galil for $k \geq 4$.

Theorem 2 ([14, 20]). *For $k \geq 3$, k-Edge Colouring is NP-complete even for k-regular graphs.*

Due to the above hardness results we may wish to restrict the input to some special graph class. A natural property of a graph class is to be closed under vertex deletion. Such graph classes are called *hereditary* and they form the focus of our paper. To give an example, bipartite graphs form a hereditary graph class, and it is well-known that they have chromatic index Δ. Hence, Edge Colouring is polynomial-time solvable for bipartite graphs, which are perfect and triangle-free. In contrast, Cai and Ellis [4] proved that for every $k \geq 3$, k-Edge Colouring is NP-complete for k-regular comparability graphs, which are also perfect. They also proved the following two results, the first one of which shows that Edge Colouring is NP-complete for triangle-free graphs (the graph C_s denotes the cycle on s vertices).

Theorem 3 ([4]). *Let $k \geq 3$ and $s \geq 3$. Then k-Edge Colouring is NP-complete for k-regular C_s-free graphs.*

Theorem 4 ([4]). *Let $k \geq 3$ be an odd integer. Then k-Edge Colouring is NP-complete for k-regular line graphs of bipartite graphs.*

It is also known that Edge Colouring is polynomial-time solvable for chordless graphs [22], series-parallel graphs [16], split-indifference graphs [26] and for graphs of treewidth at most k for any constant k [1].

It is not difficult to see that a graph class $\mathcal{G}$ is hereditary if and only if it can be characterized by a set $\mathcal{F}_{\mathcal{G}}$ of forbidden induced subgraphs (see, for example, [17]). Malyshev determined the complexity of 3-Edge Colouring for every hereditary graph class $\mathcal{G}$, for which $\mathcal{F}_{\mathcal{G}}$ consists of graphs that each have at most five vertices, except perhaps two graphs that may contain six vertices [23]. Malyshev performed a

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similar complexity study for Edge Colouring for graph classes defined by a family of forbidden (but not necessarily induced) graphs with at most seven vertices and at most six edges [24].

We focus on the case where $\mathcal{F}_\mathcal{G}$ consists of a single graph H. A graph G is *H-free* if G does not contain an induced subgraph isomorphic to H. We obtain the following dichotomy for H-free graphs.

Theorem 5. *Let $k \geq 3$ be an integer and H be a graph. If H is a linear forest, then k-*Edge Colouring *is polynomial-time solvable for H-free graphs. Otherwise k-*Edge Colouring *is* NP*-complete even for k-regular H-free graphs.*

We obtain Theorem 5 by combining Theorems 3 and 4 with two new results. In particular, we will prove a hardness result for k-regular claw-free graphs for even integers k (as Theorem 4 is only valid when k is odd).

2 Preliminaries

The graphs C_n, P_n and K_n denote the path, cycle and complete graph on n vertices, respectively. A set I is an *independent set* of a graph G if all vertices of I are pairwise nonadjacent in G. A graph G is *bipartite* if its vertex set can be partitioned into two independent sets A and B. If there exists an edge between every vertex of A and every vertex of B, then G is *complete bipartite*. The *claw* $K_{1,3}$ is the complete bipartite graph with $|A| = 1$ and $|B| = 3$.

Let G_1 and G_2 be two vertex-disjoint graphs. The *join* operation $\times$ adds an edge between every vertex of G_1 and every vertex of G_2. The *disjoint union* operation $+$ merges G_1 and G_2 into one graph without adding any new edges, that is, $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. We write rG to denote the disjoint union of r copies of a graph G.

A *forest* is a graph with no cycles. A *linear forest* is a forest of maximum degree at most 2, or equivalently, a disjoint union of one or more paths. A graph G is a *cograph* if G can be generated from K_1 by a sequence of join and disjoint union operations. A graph is a cograph if and only if it is P_4-free (see, for example, [3]). The following well-known lemma follows from this equivalence and the definition of a cograph.

Lemma 1. *Every connected P_4-free graph on at least two vertices has a spanning complete bipartite subgraph.*

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, the graph $G[S] = (S, \{uv \in E \mid u, v \in S\})$ denotes the subgraph of G *induced* by S. We say that S is *connected* if $G[S]$ is connected. Recall that a graph G is *H-free* for some graph H if G does not contain H as an induced subgraph. A subset $D \subset V(G)$ is *dominating* if every vertex of $V(G) \setminus D$ is adjacent to least one vertex of D. We will need the following result of Camby and Schaudt.

Theorem 6 ([5]). *Let $t \geq 4$ and G be a connected P_t-free graph. Let X be any minimum connected dominating set of G. Then $G[X]$ is either P_{t-2}-free or isomorphic to P_{t-2}.*

Let $G = (V, E)$ be some graph. The *degree* of a vertex $u \in V$ is equal to the size of its neighbourhood $N(u) = \{v \mid uv \in E\}$. The graph G is *r-regular* if every vertex of G has degree r. The *line graph* of G is the graph $L(G)$, which has vertex set E and an edge between two distinct vertices e and f if and only if e and f have a common end-vertex in G.

3 The Proof of Theorem 5

To prove our dichotomy, we first consider the case where the forbidden induced subgraph H is a claw. As line graphs are claw-free, we only need to deal with the case where the number of colours k is even due to Theorem 4. For proving this case we need another result of Cai and Ellis, which we will use as a lemma. Let c be a k-edge colouring of a graph $G = (V, E)$. Then a vertex $u \in V$ *misses* colour i if none of the edges incident to u is coloured i.

Lemma 2 ([4]). *For even $k \geq 2$, the complete graph K_k has a k-edge colouring with the property that $V(K_k)$ can be partitioned into sets $\{u_i, u_i'\}$ $(1 \leq i \leq \frac{k}{2})$, such that for $i = 1, \ldots, \frac{k}{2}$, vertices u_i and u_i' miss the same colour, which is not missed by any of the other vertices.*

We use Lemma 2 to prove the following result, which solves the case where k is even and $H = K_{1,3}$.

Lemma 3. *Let $k \geq 4$ be an even integer. Then k-*Edge Colouring *is* NP*-complete for k-regular claw-free graphs.*

Proof. Recall that k-Edge Colouring for k-regular graphs is NP-complete for every integer $k \geq 4$ due to Theorem 2. Consider an instance (G, k) of k-Edge Colouring, where G is k-regular for some even integer $k = 2\ell \geq 4$. From G we construct a graph G' as follows. First we replace every vertex v in G by the gadget $H(v)$ shown in Figure 1. Next we connect the different gadgets in the following way. Every gadget $H(v)$ has exactly k pendant edges, which are incident with vertices $v_1, \ldots, v_\ell, v_{\ell+1}, \ldots, v_{2\ell}$, respectively. As G is k-regular, every vertex has k neighbours in G. Hence, we can identify each edge uv of G with a unique edge $u_h v_i$ in G', which is a pendant edge of both $H(u)$ and $H(v)$. It is readily seen that G' is k-regular and claw-free.

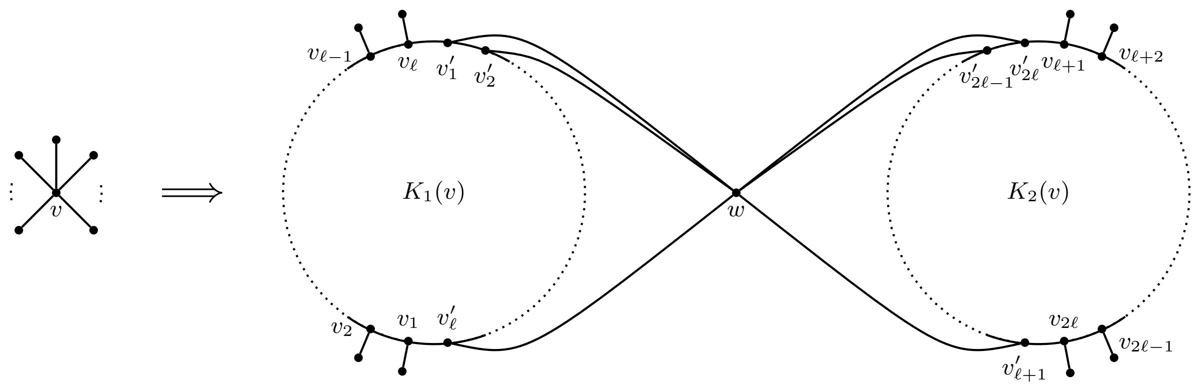

Fig. 1. The gadget $H(v)$ where $K_i(v)$ is a complete graph of size 2ℓ for $i = 1, 2$. Note that edges inside $K_1(v)$ and $K_2(v)$ are not drawn.

First suppose that G is k-edge colourable. Let c be a k-edge colouring of G. Consider a vertex $v \in V(G)$. For every neighbour u of v in G, we colour the pendant edge in $H(v)$ corresponding to the edge uv with colour $c(uv)$. As c assigned different colours to the edges incident to v, the 2ℓ pendant edges of $H(v)$ will receive pairwise distinct colours, which we denote by $x_1, \ldots, x_\ell, y_1, \ldots, y_\ell$. By Lemma 2, we can colour the edges of $K_1(v)$ in such a way that for $i = 1, \ldots, \ell$, v_i and v_i' miss colour x_i. For $i = 1, \ldots, \ell$, we can therefore assign colour x_i to edge $v_i'w$. Similarly, we may assume that for $i = 1, \ldots, \ell$, $v_{\ell+i}$ and $v_{\ell+i}'$ miss colour y_i. For $i = 1, \ldots, \ell$, we can therefore assign colour y_i to edge $v_{\ell+i}'w$. Recall that the colours $x_1, \ldots, x_\ell$, $y_1, \ldots, y_\ell$ are all different. Hence, doing this procedure for each vertex of G yields a k-edge colouring c' of G'.

Now suppose that G' is k-edge colourable. Let c' be a k-edge colouring of G'. Consider some $v \in V(G)$. Denote the pendant edges of $H(v)$ by e_i for $i = 1, \ldots, 2\ell$, where e_i is incident to v_i (and to some vertex u_h in a gadget $H(u)$ for each neighbour u of v in G). Suppose that c' gave colour x to an edge wv_i' for some $1 \leq i \leq \ell$, say to wv_1', but not to any edge e_i for $i = 1, \ldots, \ell$. Note that $wv_2', \ldots, wv_\ell'$ cannot be coloured x. As every vertex of G' has degree $k = 2\ell$, every v_i with $1 \leq i \leq \ell$ and every v_j' with $2 \leq j \leq \ell$ is incident to some edge coloured x. As x is neither the colour of $e_1, \ldots, e_\ell$ nor the colour of $wv_2', \ldots, wv_\ell'$, the complete graph $K_1(v) - v_1'$ contains a perfect matching all of whose edges have colour x. However, $K_1(v) - v_1'$ has odd size $2\ell - 1$. Hence, this is not possible. We conclude that each of the (pairwise distinct) colours of $wv_1', \ldots, wv_\ell'$, which we denote by $x_1, \ldots, x_\ell$, is the colour of an edge e_i for some $1 \leq i \leq \ell$.

Let $y_1, \ldots, y_\ell$ be the (pairwise distinct) colours of $wv'_{\ell+1}, \ldots, wv'_{2\ell}$, respectively. By the same arguments as above, we find that each of those colours is also the colour of a pendant edge of $H(v)$ that is incident to a vertex $v_{\ell+i}$ for some $1 \le i \le \ell$. Note that $x_1, \ldots, x_\ell$, $y_1, \ldots, y_\ell$ are 2ℓ pairwise distinct colours, as they are colours of edges incident to the same vertex, namely vertex w. Hence, we can define a k-colouring c of G by setting $c(uv) = c'(u_h v_i)$ for every edge $uv \in E(G)$ with corresponding edge $u_h v_i \in E(G')$. □

We note that the graph G' in the proof of Lemma 3 is not a line graph, as the gadget $H(v)$ is not a line graph: the vertices v'_1, v'_2, v_1, w form a diamond and by adding the pendant edge incident to v_1 and the edge $wv'_{\ell+1}$ we obtain an induced subgraph of $H(v)$ that is not a line graph.

To handle the case where the forbidden induced subgraph H is a path, we make the following observation.

Observation 1 *If a graph G of maximum degree k has a dominating set of size at most p, then G has at most $p(k+1)$ vertices.*

We use Observation 1 to prove the following lemma.

Lemma 4. *Let $k \ge 0$ and $t \ge 1$. Every connected P_t-free graph of maximum degree k has at most $f(k,t)$ vertices for some function f that only depends on k and t.*

Proof. Let G be a connected P_t-free graph of maximum degree at most k. We use induction on t.

First suppose $t = 4$ (and observe that if the claim holds for $t = 4$, it also holds for $t \le 3$). As G is connected, G has a dominating set of size 2 due to Lemma 1. Hence, by Observation 1, we find that G has at most $f(k,2) = 2(k+1)$ vertices.

Now suppose $t \ge 5$. Let X be an arbitrary minimum connected dominating set of G. By Theorem 6, $G[X]$ is either P_{t-2}-free or isomorphic to P_{t-2}. In the first case we use the induction hypothesis to conclude that $G[X]$ has at most $f(k,t-2)$ vertices. Hence, G has at most $f(k,t-2)(k+1)$ vertices by Observation 1. In the second case, we find that G has at most $(t-2)(k+1)$ vertices. We set $f(k,t) = \max\{f(k,t-2)(k+1), (t-2)(k+1)\}$. □

We use Lemma 4 to prove our next lemma.

Lemma 5. *Let $k \ge 3$ and $t \ge 1$. Then k-*Edge Colouring *is linear-time solvable for P_t-free graphs.*

Proof. Let G be a P_t-free graph. We compute the set of connected components of G in linear time. For each connected component D of G we do as follows. We first compute in linear time the maximum degree Δ_D of D. If $\Delta_D \le k-1$, then D is k-edge colourable by Theorem 1. If $\Delta_D \ge k+1$, then D is not k-edge colourable. Hence, we may assume that $\Delta_D = k$. By Lemma 4, D has at most $f(k,t)$ vertices for some function f that only depends on k and t. As we assume that k and t are constants, this means that we can now check in constant time if D is k-edge colourable. Note that G is k-edge colourable if and only if every connected component of G is k-edge colourable. Hence, by using the above procedure, deciding if G is k-edge colourable takes linear time. □

We are now ready to prove Theorem 5, which we restate below.

Theorem 5. (restated) *Let $k \ge 3$ be an integer and H be a graph. If H is a linear forest, then k-*Edge Colouring *is linear-time solvable for H-free graphs. Otherwise k-*Edge Colouring *is* NP*-complete even for k-regular H-free graphs.*

Proof. First suppose that H contains a cycle C_s for some $s \ge 3$. Then the class of H-free graphs is a superclass of the class of C_s-free graphs. This means that we can apply Theorem 3. From now on assume that H contains no cycle, so H is a forest. Suppose that H contains a vertex of degree at least 3. Then the class of H-free graphs is a superclass of the class of $K_{1,3}$-free graphs, which in turn forms a superclass of the class of line graphs. Hence, if k is odd, then we apply Theorem 4, and if k is even, then we apply Lemma 3. From now on assume that H contains no cycle and no vertex of degree at least 3. Then H is a linear forest, say with ℓ connected components. Let $t = \ell|V(H)|$. Then the class of H-free graphs is contained in the class of P_t-free graphs. Hence we may apply Lemma 5. This completes the proof of Theorem 5. □

4 Conclusions

We gave a complete complexity classification of k-Edge Colouring for H-free graphs, showing a dichotomy between linear-time solvable cases and NP-complete cases. We saw that this depends on H being a linear forest or not. It would be interesting to prove a dichotomy result for Edge Colouring restricted to H-free graphs. Note that due to Theorem 5 we only need to consider the case where H is a linear forest. However, even determining the complexity for small linear forests H, such as the cases where $H = 2P_2$ and $H = P_4$, turns out to be a difficult problem. In fact, the computational complexity of Edge Colouring for split graphs, or equivalently, $(2P_2, C_4, C_5)$-free graphs [10] and for P_4-free graphs has yet to be settled, despite the efforts towards solving the problem for these graph classes [6, 8, 21].

On a side note, a graph is k-edge colourable if and only if its line graph is k-vertex colourable. In contrast to the situation for Edge Colouring, the computational complexity of Vertex Colouring has been fully classified for H-free graphs [19]. However, the computational complexity for k-Vertex Colouring restricted to H-free graphs has not been fully classified. It is known that for every $k \geq 3$, k-Vertex Colouring on H-free graphs is NP-complete if H contains a cycle [9] or an induced claw [14, 20], but the case where H is a linear forest has not been settled yet. The complexity status of k-Vertex Colouring is even still open for P_t-free graphs. More precisely, it is known that the cases $k \leq 2$, $t \geq 1$ (trivial), $k \geq 3$, $t \leq 5$ [13], $k = 3$, $6 \leq t \leq 7$ [2] and $k = 4$, $t = 6$ [7] are polynomial-time solvable and that the cases $k = 4$, $t \geq 7$ [15] and $k \geq 5$, $t \geq 6$ [15] are NP-complete. However, the remaining cases, that is, the cases where $k = 3$ and $t \geq 8$ are still open. We refer to the survey [11] or some recent papers [12, 18, 25] for further background information.

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