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The Duality Theorem

Every maximization LP problem in the standard form gives rise to a minimization LP problem called the dual problem. The two problems are linked in an interesting way. Every feasible solution in one yields a bound on the optimal value of the other. In fact, if one of the two problems has an optimal solution, then so does the other, and the two optimal values coincide. This fact, known as the Duality Theorem, is the subject of the present chapter. We shall also note that, in managerial applications, the variables featured in the dual problem can be interpreted in a very useful way.

MOTIVATION: FINDING UPPER BOUNDS ON THE OPTIMAL VALUE

We shall begin this chapter with the following LP problem:

$$\begin{array}{lrcl}
\text{maximize} & 4x_1 + x_2 + 5x_3 + 3x_4 & & \\
\text{subject to} & x_1 - x_2 - x_3 + 3x_4 & \leq & 1 \\
 & 5x_1 + x_2 + 3x_3 + 8x_4 & \leq & 55 \\
 & -x_1 + 2x_2 + 3x_3 - 5x_4 & \leq & 3 \\
 & x_1, x_2, x_3, x_4 & \geq & 0.
\end{array}$$

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Rather than *solving* it, we shall try to get a quick *estimate* of the optimal value z^* of its objective function. To get a reasonably good lower bound on z^*, we need only come up with a reasonably good feasible solution. For example, the bound $z^* \geq 5$ comes from considering the feasible solution $(0, 0, 1, 0)$. The feasible solution $(2, 1, 1, \frac{1}{3})$ shows that $z^* \geq 15$. Better yet, the feasible solution $(3, 0, 2, 0)$ yields $z^* \geq 22$. Needless to say, such guesswork is vastly inferior to the systematic attack by the simplex method: even if we were lucky enough to hit on the optimal solution, our guess would provide no *proof* that the solution is indeed optimal.

We shall not pursue this line any further: the subject of this chapter stems from a similar quest for *upper* bounds on z^*. For example, a glance at the data suggests that $z^* \leq \frac{275}{3}$. Indeed, multiplying the second constraint by $\frac{5}{3}$ we obtain the inequality

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

Hence every feasible solution (x_1, x_2, x_3, x_4) satisfies the inequality

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}.$$

In particular, this inequality holds for the optimal solution and so $z^* \leq \frac{275}{3}$. With a little inspiration, we can improve this bound considerably. For instance, the sum of the second and third constraints reads

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58.$$

Therefore, $z^* \leq 58$. Rather than searching for further improvements in a haphazard way, we shall now describe the strategy in precise and general terms.

We construct *linear combinations* of the constraints. That is, we multiply the first constraint by some number y_1, the second by y_2, the third by y_3, and then we add them up. (In the first case, we had $y_1 = 0, y_2 = \frac{5}{3}, y_3 = 0$; in the second case, we had $y_1 = 0, y_2 = y_3 = 1$.) The resulting inequality reads

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \\ \leq y_1 + 55y_2 + 3y_3. \tag{5.1}$$

Of course, each of the three multipliers y_i must be nonnegative: otherwise the corresponding inequality would reverse its direction. Next, we want to use the left-hand side of (5.1) as an upper bound on $z = 4x_1 + x_2 + 5x_3 + 3x_4$. This can be justified only if in (5.1), the coefficient at each x_j is at least as big as the corresponding coefficient in z. More explicitly, we want

$$\begin{aligned} y_1 + 5y_2 - y_3 &\geq 4 \\ -y_1 + y_2 + 2y_3 &\geq 1 \\ -y_1 + 3y_2 + 3y_3 &\geq 5 \\ 3y_1 + 8y_2 - 5y_3 &\geq 3. \end{aligned}$$

If the multipliers y_i are nonnegative and if they satisfy these four inequalities, then we may safely conclude that every feasible solution (x_1, x_2, x_3, x_4) satisfies the inequality

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3.$$

In particular, this inequality is satisfied by the optimal solution; therefore

$$z^* \leq y_1 + 55y_2 + 3y_3.$$

Of course, we want as small an upper bound on z^* as we can possibly get. Thus, we are led to the following LP problem:

$$\begin{array}{lrcrcrl}
\text{minimize} & y_1 & + & 55y_2 & + & 3y_3 & \\
\text{subject to} & y_1 & + & 5y_2 & - & y_3 & \geq 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq 3 \\
 & & & & & y_1, y_2, y_3 & \geq 0.
\end{array}$$

THE DUAL PROBLEM

This problem is called the *dual* of the original one; the original problem is called the *primal* problem. In general, the dual of the problem

$$\begin{array}{lll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots, m) \\
 & x_j \geq 0 & (j = 1, 2, \ldots, n)
\end{array} \tag{5.2}$$

is defined to be the problem

$$\begin{array}{lll}
\text{minimize} & \sum_{i=1}^{m} b_i y_i & \\
\text{subject to} & \sum_{i=1}^{m} a_{ij} y_i \geq c_j & (j = 1, 2, \ldots, n) \\
 & y_i \geq 0 & (i = 1, 2, \ldots, m).
\end{array} \tag{5.3}$$

(Note that the dual of a maximization problem is a minimization problem. Furthermore, the m primal constraints $\sum a_{ij} x_j \leq b_i$ are in a one-to-one correspondence with the m dual variables y_i; conversely, the n dual constraints $\sum a_{ij} y_i \geq c_j$ are in a one-to-one correspondence with the n primal variables x_j. The coefficient at each variable in the objective function, primal or dual, appears in the other problem as the right-hand side of the corresponding constraint.)

As in our example, every feasible solution of the dual yields an upper bound on the optimal value of the primal. More explicitly, for every primal feasible solution $(x_1, x_2, \ldots, x_n)$ and for every dual feasible solution $(y_1, y_2, \ldots, y_m)$ we have

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i. \tag{5.4}$$

The proof of (5.4), which was illustrated at the beginning of this section, can be written down succinctly as

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i.$$

Inequality (5.4) is extremely useful: if we happen to stumble across a primal feasible solution $(x_1^*, x_2^*, \ldots, x_n^*)$ and a dual feasible solution $(y_1^*, y_2^*, \ldots, y_m^*)$ such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then we may conclude that both of these solutions are optimal. Indeed, (5.4) implies that every primal feasible solution $(x_1, x_2, \ldots, x_n)$ satisfies

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^*$$

and that every dual feasible solution $(y_1, y_2, \ldots, y_m)$ satisfies

$$\sum_{i=1}^{m} b_i y_i \geq \sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

For instance, we have an easy way of showing that the primal feasible solution $x_1 = 0$, $x_2 = 14$, $x_3 = 0$, $x_4 = 5$ of our original example is optimal: just consider the dual feasible solution $y_1 = 11$, $y_2 = 0$, $y_3 = 6$. It is not at all obvious, however, that an analogous proof of optimality can be given for *every* LP problem that has an optimal solution; this fact is the central theorem of linear programming.

THE DUALITY THEOREM AND ITS PROOF

The explicit version of the theorem comes from D. Gale, H. W. Kuhn, and A. W. Tucker (1951); its notions originated in conversations between G. B. Dantzig and J. von Neumann in the fall of 1947.

THEOREM 5.1 (The Duality Theorem). If the primal (5.2) has an optimal solution $(x_1^*, x_2^*, \ldots, x_n^*)$, then the dual (5.3) has an optimal solution $(y_1^*, y_2^*, \ldots, y_m^*)$ such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*. \tag{5.5}$$

Before presenting the proof, let us briefly illustrate its crucial point: the optimal solution of the *dual* problem can be read off the z-row of the final dictionary for the *primal* problem. In the example that we used to motivate the concept of the dual problem, the final dictionary reads

$$\begin{array}{rcl} x_2 &=& 14 - 2x_1 - 4x_3 - 5x_5 - 3x_7 \\ x_4 &=& 5 - x_1 - x_3 - 2x_5 - x_7 \\ x_6 &=& 1 + 5x_1 + 9x_3 + 21x_5 + 11x_7 \\ \hline z &=& 29 - x_1 - 2x_3 - 11x_5 - 6x_7. \end{array}$$

Note that the slack variables x_5, x_6, x_7 can be matched up with the dual variables y_1, y_2, y_3 in a natural way: for instance, x_5 is the slack variable in the first constraint, whereas y_1 represents the multiplier for the same constraint. By the same logic, x_6 goes with y_2 and x_7 goes with y_3. In the *z-row* of the dictionary, the coefficients at the slack variables are

$$-11 \text{ at } x_5, \quad 0 \text{ at } x_6, \quad -6 \text{ at } x_7.$$

Assigning these values with reversed signs to the corresponding dual variables, we obtain the desired optimal solution of the dual:

$$y_1 = 11, \quad y_2 = 0, \quad y_3 = 6.$$

At first, this may seem like pulling a rabbit out of a hat; however, the following general argument explains the magic.

PROOF OF THEOREM 5.1. We need only find a *feasible* solution $(y_1^*, y_2^*, \ldots, y_m^*)$ satisfying (5.5); indeed, such a solution will be *optimal* by virtue of the remarks following (5.4). In order to find that solution, we solve the primal problem by the simplex method; having introduced the slack variables

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m) \tag{5.6}$$

we eventually arrive at the final dictionary. For the sake of definiteness, let us say that the last row of that dictionary reads

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k x_k. \tag{5.7}$$

In (5.7), each $\bar{c}_k$ is a nonpositive number (in fact, $\bar{c}_k = 0$ whenever x_k is a basic variable). In addition, z^* is the optimal value of the objective function, and so

$$z^* = \sum_{j=1}^{n} c_j x_j^*. \tag{5.8}$$

Defining

$$y_i^* = -\bar{c}_{n+i} \qquad (i = 1, 2, \ldots, m) \tag{5.9}$$

we claim that $(y_1^*, y_2^*, \ldots, y_m^*)$ is a dual feasible solution satisfying (5.5); the rest of the proof consists of a straightforward verification of our claim. Substituting $\sum c_j x_j$ for z and substituting from (5.6) for the slack variables in (5.7) we obtain the identity

$$\sum_{j=1}^{n} c_j x_j = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)$$

which may be written as

$$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} b_i y_i^* \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j.$$

This identity, having been obtained by algebraic manipulations from the definitions of the slack variables and the objective function, must hold for every choice of values of $x_1, x_2, \ldots, x_n$. Hence we have

$$z^* = \sum_{i=1}^{m} b_i y_i^* \tag{5.10}$$

and

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \qquad (j = 1, 2, \ldots, n). \tag{5.11}$$

Since $\bar{c}_k \leq 0$ for every $k = 1, 2, \ldots, n + m$, (5.11) and (5.9) imply

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \qquad (j = 1, 2, \ldots, n)$$

$$y_i^* \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Finally, (5.10) and (5.8) imply (5.5). ■

RELATIONSHIP BETWEEN THE PRIMAL AND DUAL PROBLEMS

Next, let us point out that the dual of the dual is always the primal problem. Indeed, the dual problem may be written as

$$\begin{aligned}
&\text{maximize} && \sum_{i=1}^{m} (-b_i)y_i \\
&\text{subject to} && \sum_{i=1}^{m} (-a_{ij})y_i \leq -c_j \qquad (j = 1, 2, \ldots, n) \\
& && y_i \geq 0 \qquad (i = 1, 2, \ldots, m).
\end{aligned}$$

The dual of this problem is

$$\begin{aligned}
&\text{minimize} && \sum_{j=1}^{n} (-c_j)x_j \\
&\text{subject to} && \sum_{j=1}^{n} (-a_{ij})x_j \geq -b_i \qquad (i = 1, 2, \ldots, m) \\
& && x_j \geq 0 \qquad (j = 1, 2, \ldots, n)
\end{aligned}$$

which is clearly equivalent to the original problem. A nice corollary to this observation and to the duality theorem is that the primal problem has an optimal solution *if and only if* the dual problem has an optimal solution. Note also that if the primal is unbounded, then the dual must be infeasible [this follows directly from (5.4)]. By the same argument, if the dual is unbounded then the primal must be infeasible. However, both primal and dual may be infeasible at the same time. For example, both the problem

$$\begin{aligned}
&\text{maximize} && 2x_1 - x_2 \\
&\text{subject to} && x_1 - x_2 \leq 1 \\
& && -x_1 + x_2 \leq -2 \\
& && x_1, x_2 \geq 0
\end{aligned}$$

and its dual are infeasible. These conclusions are summarized in Table 5.1.

Table 5.1 Primal–Dual Combinations

		Dual		
		Optimal	Infeasible	Unbounded
Primal	Optimal	Possible	Impossible	Impossible
	Infeasible	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

In particular, if the primal problem has a feasible solution *and* if the dual problem has a feasible solution, then both problems have optimal solutions.

Duality has important practical implications. In certain cases, we may find it advantageous to apply the simplex method to the dual of the problem that we are really interested in. (Of course, the optimal solution of the primal problem can then be read directly off the final dictionary for the dual.) For example, if $m = 99$ and $n = 9$, then dictionaries will have 100 rows in the primal problem but only 10 rows in the dual. Since the typical number of simplex iterations is proportional to the number of rows in a dictionary and relatively insensitive to the number of variables, we shall most likely be better off solving the dual problem.

From a theoretical point of view, duality is important because it points out an elegant and succinct way of proving optimality of solutions of LP problems: as we have observed, an optimal solution of the dual problem provides a "certificate of optimality" for an optimal solution of the primal problem, and vice versa. Furthermore, the duality theorem asserts that for *every* optimal solution there is a certificate of optimality. To appreciate the impact of this fact, consider a student who is supposed to solve the problem

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \ldots, m) \\ & x_j \ge 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{5.12}$$

Applying the simplex method to (5.12), the student finds simultaneously an optimal solution $x_1^*, x_2^*, \ldots, x_n^*$ of (5.12) and an optimal solution $y_1^*, y_2^*, \ldots, y_m^*$ of the dual problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j \qquad (j = 1, 2, \ldots, n) \\ & y_i \ge 0 \qquad (i = 1, 2, \ldots, m). \end{aligned} \tag{5.13}$$

Then he shows *both* solutions to his supervisor. The supervisor has an easy way of checking the correctness of the answer. To check the *feasibility* of the allegedly optimal solution, she has to verify the inequalities

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j^* &\le b_i \qquad (i = 1, 2, \ldots, m) \\ x_j^* &\ge 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{5.14}$$

To check its *optimality*, she has to verify the inequalities

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \qquad (j = 1, 2, \ldots, n) \tag{5.15}$$
$$y_i^* \geq 0 \qquad (i = 1, 2, \ldots, m)$$

and the equation

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*. \tag{5.16}$$

Of course, the computational effort involved in these *verifications* is much smaller than the computational effort required to *solve* (5.12) from scratch by the simplex method.

COMPLEMENTARY SLACKNESS

Now we shall show how the supervisor can often recover the certificate of optimality $y_1^*, y_2^*, \ldots, y_m^*$ from the optimal solution $x_1^*, x_2^*, \ldots, x_n^*$ alone. The key to the procedure is a convenient way of breaking down equation (5.16) into simple constituents.

THEOREM 5.2. Let $x_1^*, x_2^*, \ldots, x_n^*$ be a feasible solution of (5.12) and let $y_1^*, y_2^*, \ldots, y_m^*$ be a feasible solution of (5.13). Necessary and sufficient conditions for simultaneous optimality of $x_1^*, x_2^*, \ldots, x_n^*$ and $y_1^*, y_2^*, \ldots, y_m^*$ are

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \quad \text{or} \quad x_j^* = 0 \quad \text{(or both)} \quad \text{for every} \quad j = 1, 2, \ldots, n \tag{5.17}$$

and

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \quad \text{or} \quad y_i^* = 0 \quad \text{(or both)} \quad \text{for every} \quad i = 1, 2, \ldots, m. \tag{5.18}$$

PROOF. Assumptions (5.14) and (5.15) imply

$$c_j x_j^* \leq \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* \qquad (j = 1, 2, \ldots, n) \tag{5.19}$$

$$\left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \leq b_i y_i^* \qquad (i = 1, 2, \ldots, m) \tag{5.20}$$

and so

$$\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \leq \sum_{i=1}^{m} b_i y_i^*. \tag{5.21}$$

Hence, (5.21) holds with equalities throughout if and only if equalities hold in (5.19) and (5.20). One way to guarantee the equality $c_j x_j^* = (\sum a_{ij} y_i^*) x_j^*$ is to insist that $x_j^* = 0$; failing that, we must require $c_j = \sum a_{ij} y_i^*$. Hence, equalities hold in (5.19) if and only if condition (5.17) is satisfied. Similarly, equalities hold in (5.20) if and only if condition (5.18) is satisfied.

To summarize, conditions (5.17) and (5.18) are necessary and sufficient for (5.16) to hold. On the other hand, the Duality Theorem shows that (5.16) is necessary and sufficient for simultaneous optimality of $x_1^*, x_2^*, \ldots, x_n^*$ and $y_1^*, y_2^*, \ldots, y_m^*$. The proof is completed. ■

Conditions (5.17) and (5.18) gain simplicity as soon as we introduce the slack variables

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m)$$

$$y_{m+j} = -c_j + \sum_{i=1}^{m} a_{ij} y_i \qquad (j = 1, 2, \ldots, n).$$

As we observed once before, the primal slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ are naturally matched up with the dual decision variables $y_1, y_2, \ldots, y_m$: each variable x_{n+i} denotes the slack in the ith primal constraint, whereas the corresponding y_i represents the multiplier at the same constraint. Similarly, each primal decision variable x_j is matched with the dual slack y_{m+j}. Conditions (5.17) and (5.18) require that in each of the $m + n$ matching pairs, at least one variable must have value zero. These conditions are usually called the *complementary slackness* conditions; Theorem 5.2 itself is referred to as the Complementary Slackness Theorem.

It is an easy task to convert Theorem 5.2 into a form in which its applicability becomes evident.

THEOREM 5.3. A feasible solution $x_1^*, x_2^*, \ldots, x_n^*$ of (5.12) is optimal if and only if there are numbers $y_1^*, y_2^*, \ldots, y_m^*$ such that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i^* &= c_j \quad \text{whenever} \quad x_j^* > 0 \\ y_i^* &= 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_{ij} x_j^* < b_i \end{aligned} \tag{5.22}$$

and such that

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i^* &\ge c_j \quad \text{for all} \quad j = 1, 2, \ldots, n \\ y_i^* &\ge 0 \quad \text{for all} \quad i = 1, 2, \ldots, m. \end{aligned} \tag{5.23}$$

PROOF. If $x_1^*, x_2^*, \ldots, x_n^*$ is optimal, then, by Theorem 5.1, there is an optimal solution $y_1^*, y_2^*, \ldots, y_m^*$ of (5.13). That solution, being feasible, satisfies (5.23). By Theorem 5.2, the two optimal solutions satisfy the complementary slackness conditions (5.22).

Conversely, if $y_1^*, y_2^*, \ldots, y_m^*$ satisfy (5.23), then they constitute a feasible solution of (5.13). If they satisfy (5.22) as well, then, by Theorem 5.2, $x_1^*, x_2^*, \ldots, x_n^*$ is an optimal solution of (5.12) and $y_1^*, y_2^*, \ldots, y_m^*$ is an optimal solution of (5.13). ∎

Theorem 5.3 is often useful in checking the optimality of allegedly optimal solutions when no certificate of optimality is provided. Confronted with an allegedly optimal solution $x_1^*, x_2^*, \ldots, x_n^*$ of (5.12), we first set up the system of linear equations (5.22) and solve for $y_1^*, y_2^*, \ldots, y_m^*$. If the solution $y_1^*, y_2^*, \ldots, y_m^*$ is *unique*, then we are in business: $x_1^*, x_2^*, \ldots, x_n^*$ is optimal if and only if (5.23) holds. We shall illustrate this situation with two examples.

First, let us consider the claim that

$$x_1^* = 2, \quad x_2^* = 4, \quad x_3^* = 0, \quad x_4^* = 0, \quad x_5^* = 7, \quad x_6^* = 0$$

is an optimal solution of the problem

$$\begin{array}{lrcrcrcrcrcrcr}
\text{maximize} & 18x_1 & - & 7x_2 & + & 12x_3 & + & 5x_4 & & & + & 8x_6 & & \\
\text{subject to} & 2x_1 & - & 6x_2 & + & 2x_3 & + & 7x_4 & + & 3x_5 & + & 8x_6 & \le & 1 \\
 & -3x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & + & x_5 & + & 2x_6 & \le & -2 \\
 & 8x_1 & - & 3x_2 & + & 5x_3 & - & 2x_4 & & & + & 2x_6 & \le & 4 \\
 & 4x_1 & & & + & 8x_3 & + & 7x_4 & - & x_5 & + & 3x_6 & \le & 1 \\
 & 5x_1 & + & 2x_2 & - & 3x_3 & + & 6x_4 & - & 2x_5 & - & x_6 & \le & 5 \\
 & & & & & & & & & & x_1, x_2, x_3, x_4, x_5, & x_6 & \ge & 0.
\end{array}$$

In this case, (5.22) reads

$$\begin{array}{rcrcrcrcrcr}
2y_1^* & - & 3y_2^* & + & 8y_3^* & + & 4y_4^* & + & 5y_5^* & = & 18 \\
-6y_1^* & - & y_2^* & - & 3y_3^* & & & + & 2y_5^* & = & -7 \\
3y_1^* & + & y_2^* & & & - & y_4^* & - & 2y_5^* & = & 0 \\
 & & y_2^* & & & & & & & = & 0 \\
 & & & & & & & & y_5^* & = & 0.
\end{array}$$

Since its solution $(\frac{1}{3}, 0, \frac{5}{3}, 1, 0)$ satisfies (5.23), the proposed solution $x_1^*, x_2^*, \ldots, x_6^*$ is optimal.

Second, let us consider the claim that

$$x_1^* = 0, \quad x_2^* = 2, \quad x_3^* = 0, \quad x_4^* = 7, \quad x_5^* = 0$$

is an optimal solution of the problem

$$
\begin{array}{ll}
\text{maximize} & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5 \\
\text{subject to} & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \le 1 \\
& x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \le 1 \\
& 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \le 22 \\
& x_1, x_2, x_3, x_4, x_5 \ge 0.
\end{array}
$$

Here (5.22) becomes

$$
\begin{aligned}
-3y_1^* + 7y_2^* + 4y_3^* &= -9 \\
y_1^* - 2y_2^* + 2y_3^* &= 4 \\
y_2^* &= 0.
\end{aligned}
$$

Since its unique solution (3.4, 0, 0.3) violates (5.23), the proposed solution $x_1^*, x_2^*, \ldots, x_5^*$ is not optimal.

Of course, this straightforward strategy for verifying optimality of allegedly optimal solutions is applicable only if the system of equations (5.22) has a unique solution. The following result points out conditions under which this is always the case.

> ***THEOREM 5.4.*** If $x_1^*, x_2^*, \ldots, x_n^*$ is a nondegenerate basic feasible solution of (5.12), then (5.22) has a unique solution.

The proof of this theorem is postponed until the end of Chapter 7, where it will become an exercise (problem 7.3).

ECONOMIC SIGNIFICANCE OF DUAL VARIABLES

For many LP problems

$$
\begin{array}{lll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j & \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i & (i = 1, 2, \ldots, m) \\
& x_j \ge 0 & (j = 1, 2, \ldots, n)
\end{array}
\qquad (5.24)
$$

arising in applications, the variables $y_1, y_2, \ldots, y_m$ in the dual problem can be given a meaningful interpretation. An indication of the way these dual variables should be interpreted follows from a heuristic argument occasionally used in elementary physics and known as "dimension analysis." For instance, suppose that (5.24) is the problem of maximizing profit in a furniture manufacturing firm. Each x_j measures the level of the output of the jth product (such as desks or chairs), and each b_i specifies the available amount of the ith resource (such as wood or metal). Note that *each* a_{ij} *is expressed in units of resource* i *per unit of product* j (in fact, each a_{ij} is the amount of resource i required in making a unit of product j) and that *each* c_j *is in dollars per unit of product* j (in fact, each c_j is the net profit brought in by a unit of product j). To make the left-hand side, $\sum a_{ij}y_i$, of each dual constraint commensurate with the right-hand side c_j, *we must express each* y_i *in dollars per unit of resource* i. Thus, one is led to suspect that each y_i measures the unit worth of the ith resource. The following theorem will validate this suspicion.

THEOREM 5.5. If (5.24) has at least one nondegenerate basic optimal solution, then there is a positive ε with the following property: If $|t_i| \leq \varepsilon$ for all $i = 1, 2, \ldots, m$, then the problem

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j && \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i && (i = 1, 2, \ldots, m) \\ &&& x_j \geq 0 && (j = 1, 2, \ldots, n) \end{aligned} \tag{5.25}$$

has an optimal solution and its optimal value equals

$$z^* + \sum_{i=1}^{m} y_i^* t_i$$

with z^* standing for the optimal value of (5.24) and with $y_1^*, y_2^*, \ldots, y_m^*$ standing for the optimal solution of its dual.

We postpone the proof of this theorem also until the end of Chapter 7, where it will become an exercise (problem 7.4). At this moment, let us note only that the uniqueness of $y_1^*, y_2^*, \ldots, y_m^*$ is guaranteed by Theorems 5.4 and 5.2.

Theorem 5.5 reveals the effects of small variations in the supplies of the resources on the total net profit of the firm. With each extra unit of resource i, the profit increases by y_i^* dollars. Hence, y_i^* specifies the maximum amount that the firm should be willing to pay, over and above the present trading price, for each extra unit of resource i. For this reason, y_i^* is often called the *marginal value* of the ith resource, the adjective "marginal" referring to the difference between the trading price and the actual worth of the resource. Another term commonly used for y_i^* in this context is the *shadow price* of the ith resource.

To illustrate these findings, imagine a forester who has 100 acres of hardwood timber. Felling the hardwood and letting the area regenerate would cost \$10 per acre in immediate resources and bring a subsequent return of \$50 per acre. An alternative course of action is to fell the hardwood and plant the area with pine; that would cost \$50 per acre with a subsequent return of \$120 per acre. Hence, the *net* profits resulting from the two treatments are \$40 and \$70 per acre, respectively. Unfortunately, the more profitable second treatment cannot be applied to the entire area since only \$4,000 is available to meet the immediate costs. Clearly, the forester's problem is to

$$\begin{array}{ll} \text{maximize} & 40x_1 + 70x_2 \\ \text{subject to} & x_1 + x_2 \le 100 \\ & 10x_1 + 50x_2 \le 4{,}000 \\ & x_1, x_2 \ge 0. \end{array}$$

Its optimal solution is $x_1^* = 25$ and $x_2^* = 75$. Hence, the forester should fell the hardwood throughout the entire area, letting 25 acres regenerate and planting the remaining 75 acres with pine. According to this program, the initial investment of \$4,000 yields the ultimate net profit of \$6,250.

Evidently, the forester's initial capital represents a valuable resource. In fact, the forester might be well advised to increase the level of this resource by taking out a short-term loan; the resulting extra profit might make up even for a drastic interest rate. For example, suppose that she could borrow \$100 now and pay back \$180 later; should she do that? On the other hand, she might be tempted to divert some of her \$4,000 to other lucrative enterprises. For example, suppose she could invest \$100 now and collect \$180 later; should she do that? According to Theorem 5.5, the answers lie hidden in the optimal solution

$$y_1^* = 32.5, \quad y_2^* = 0.75$$

of the dual problem: the forester should take out (limited) loans if and only if the interest is lower than 75 cents per dollar and she should make (small) investments if and only if the profit is greater than 75 cents per dollar.

These claims, whose validity is guaranteed by Theorem 5.5, are easy to justify directly. Having borrowed t dollars, the forester aims to

$$\begin{array}{ll} \text{maximize} & 40x_1 + 70x_2 \\ \text{subject to} & x_1 + x_2 \le 100 \\ & 10x_1 + 50x_2 \le 4{,}000 + t \\ & x_1, x_2 \ge 0. \end{array} \tag{5.26}$$

Every feasible solution x_1, x_2 of this problem satisfies the inequalities

$$\begin{aligned} 40x_1 + 70x_2 &= 32.5(x_1 + x_2) + 0.75(10x_1 + 50x_2) \\ &\leq 3{,}250 + 0.75(4{,}000 + t) = 6{,}250 + 0.75t \end{aligned} \tag{5.27}$$

and so the extra profit will never exceed $0.75t$. In fact, if $t \leq 1{,}000$, then the forester can realize the additional profit of $0.75t$ by letting

$$x_1 = 25 - 0.025t, \quad x_2 = 75 + 0.025t. \tag{5.28}$$

Investments in other enterprises give rise to negative values of t in (5.26); as a result of such investments, the net profit from the original enterprise diminishes. If $-t$ dollars are diverted to alternative investments ($-t$ is positive!) then, by (5.27), the profit from the hardwood felling enterprise will drop by $0.75(-t)$ or even more. In fact, if $-t \leq 3{,}000$, then the drop can be limited to only $0.75(-t)$ by choosing x_1 and x_2 according to (5.28).

It should perhaps be emphasized that Theorem 5.5 deals with *small* changes t_i in the resource levels; its conclusions may fail when the t_i's are large. For instance, our forester has no use for loans exceeding \$1,000 and, should she wish to invest all of her \$4,000 in another enterprise, she would be ill advised to demand only 75 cents of profit on each dollar. (A part of Theorem 5.5 can be salvaged even if the t_i's are large; see problem 5.9.)

Now suppose that a previously unavailable opportunity arises for the forester to engage in an activity such as, say, felling the hardwood and planting the area with conifer. For a quick assessment of this activity, the forester may appeal to the marginal values of her resources: \$32.5 per acre of hardwood and \$0.75 per dollar of capital. If the new activity requires a dollars per acre, then the resources consumed by this activity per acre are valued at \$(32.5 + 0.75a) and the activity is worth considering if and only if its net profit per acre exceeds this figure. Further examples of this kind are presented in problems 5.6 and 5.7.

In closing, let us mention that models of economy often fall into the realm of linear programming. In particular, many theorems concerning economic equilibria may be deduced from the Duality Theorem and the Complementary Slackness Theorem. Their discussion exceeds the scope of this text; the interested reader is referred to D. Gale (1960) and R. Dorfman, P. A. Samuelson, and R. M. Solow (1958).

PROBLEMS

5.1 Illustrate Theorem 5.1 on each of the three LP problems in problem 2.1.

△**5.2** Maximize $-x_1 - 2x_2$

subject to

$$\begin{aligned}
-3x_1 + x_2 &\le -1\\
x_1 - x_2 &\le 1\\
-2x_1 + 7x_2 &\le 6\\
9x_1 - 4x_2 &\le 6\\
-5x_1 + 2x_2 &\le -3\\
7x_1 - 3x_2 &\le 6\\
x_1, x_2 &\ge 0.
\end{aligned}$$

△**5.3** For each of the two problems below, use Theorem 5.3 to check the optimality of the proposed solution.

a. Maximize $7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$

subject to

$$\begin{aligned}
x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 &\le 4\\
4x_1 + 2x_2 - 2x_3 + x_4 + x_5 &\le 3\\
2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 &\le 5\\
3x_1 + x_2 + 2x_3 - x_4 - 2x_5 &\le 1\\
x_1, x_2, x_3, x_4, x_5 &\ge 0.
\end{aligned}$$

Proposed solution: $x_1^* = 0,\quad x_2^* = \frac{4}{3},\quad x_3^* = \frac{2}{3},\quad x_4^* = \frac{5}{3},\quad x_5^* = 0.$

b. Maximize $4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$

subject to

$$\begin{aligned}
x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\le 1\\
5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\le 4\\
4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\le 4\\
-x_2 + 2x_4 + x_5 - 5x_6 &\le 5\\
-2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\le 7\\
2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\le 5\\
x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0.
\end{aligned}$$

Proposed solution: $x_1 = 0,\quad x_2 = 0,\quad x_3 = \frac{5}{2},\quad x_4 = \frac{7}{2},\quad x_5 = 0,\quad x_6 = \frac{1}{2}.$

△**5.4** In problem 1.6, one of the possible strategies is as follows:

- Smoke all 400 bellies on regular time.
- Smoke 20 picnics on regular time and 210 on overtime.
- Smoke 40 hams on overtime and sell 440 fresh.

Use Theorem 5.3 to find out whether this strategy is optimal or not.

5.5 In problem 1.7, one of the possible strategies is as follows:

- Blend 3,754 barrels of alkylate, 2,666 barrels of catalytic, 920 barrels of straight-run, and 543 barrels of isopentane into 7,883 barrels of Avgas A.
- Blend 60 barrels of alkylate, 3,096 barrels of straight-run, and 672 barrels of isopentane into 3,828 barrels of Avgas B.
- Sell 85 barrels of isopentane raw.

Use Theorem 5.3 to find out whether this strategy is optimal or not.

5.6 In the optimal solution to problem 1.6, all the bellies and picnics are smoked. However, sufficiently drastic changes in market prices might provide an incentive to change this policy. Assume that the market price of fresh bellies increases by x dollars, while all the other prices remain fixed at their original levels. How large would x have to be in order to make it profitable for the plant to sell fresh bellies? Ask and answer a similar question for picnics. How would the sales of small amounts of fresh bellies and picnics affect the rest of the operation? What precisely do "small amounts" mean in this context?

5.7 In the optimal solution to problem 1.7, 85 barrels of isopentane are sold raw at \$4.83 per barrel. Find the break-even selling prices for raw alkylate, catalytic, and straight-run. Next, assume there is a demand for Avgas C with PN at least 80 and RVP at most 7. Find the break-even selling price of this gasoline.

5.8 Can you interpret the complementary slackness conditions in economic terms?

5.9 Let z^* be the optimal value of (5.24) and let $y_1^*, y_2^*, \ldots, y_m^*$ be any optimal solution of the dual problem. Prove that

$$\sum_{j=1}^{n} c_j x_j \le z^* + \sum_{i=1}^{m} y_i^* t_i$$

for every feasible solution $x_1, x_2, \ldots, x_n$ of (5.25).

5.10 Construct an example showing that the conclusion of Theorem 5.5 may fail if the hypothesis that (5.24) has a nondegenerate basic optimal solution is omitted.