2

How the Simplex Method Works

In this chapter, we shall learn to solve LP problems in the standard form by the simplex method. A rigorous analysis of the details will be deferred to Chapter 3.

FIRST EXAMPLE

We shall illustrate the simplex method on the following example:

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & \qquad\quad x_1, x_2, x_3 \geq 0. \end{array} \tag{2.1}$$

A preliminary step of the method consists of introducing so-called slack variables.

In order to motivate this concept, let us consider the first of our constraints,

$$2x_1 + 3x_2 + x_3 \leq 5. \tag{2.2}$$

For every feasible solution x_1, x_2, x_3, the value of the left-hand side of (2.1) is at most the value of the right-hand side; often, there may be a slack between the two values. We shall denote the slack by x_4. That is, we shall *define* $x_4 = 5 - 2x_1 - 3x_2 - x_3$; with this notation, inequality (2.2) may now be written as $x_4 \geq 0$. In an analogous way, the next two constraints give rise to variables x_5 and x_6. Finally, following a time-honored convention, we shall denote the objective function $5x_1 + 4x_2 + 3x_3$ by z. To summarize: for every *choice* of numbers x_1, x_2, and x_3, we shall *define* numbers x_4, x_5, x_6, and z by the formulas

$$\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \\
z &= \phantom{11 -{}} 5x_1 + 4x_2 + 3x_3.
\end{aligned} \tag{2.3}$$

With this notation, our problem may be restated as

$$\text{maximize} \quad z \quad \text{subject to} \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \tag{2.4}$$

The new variables x_4, x_5, x_6 defined by (2.3) are called *slack variables*; the old variables x_1, x_2, x_3 are usually referred to as the *decision variables*. It is crucial to note that the equations in (2.3) spell out an equivalence between (2.1) and (2.4). More precisely:

- Every feasible solution x_1, x_2, x_3 of (2.1) can be extended, in the unique way determined by (2.3), into a feasible solution $x_1, x_2, \ldots, x_6$ of (2.4).
- Every feasible solution $x_1, x_2, \ldots, x_6$ of (2.4) can be restricted, simply by deleting the slack variables, into a feasible solution x_1, x_2, x_3 of (2.1).
- This correspondence between feasible solutions of (2.1) and feasible solutions of (2.4) carries optimal solutions of (2.1) onto optimal solutions of (2.4), and vice versa.

The grand strategy of the simplex method is that of *successive improvements*: having found some feasible solution $x_1, x_2, \ldots, x_6$ of (2.4), we shall try to proceed to another feasible solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6$, which is better in the sense that

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5x_1 + 4x_2 + 3x_3.$$

Repeating this process a finite number of times, we shall eventually arrive at an optimal solution.

To begin with, we need some feasible solution $x_1, x_2, \ldots, x_6$. Finding one in our example presents no difficulty: setting the decision variables x_1, x_2, x_3 at zero, we evaluate the slack variables x_4, x_5, x_6 from (2.3). Hence our initial solution,

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 5, \quad x_5 = 11, \quad x_6 = 8 \tag{2.5}$$

yields $z = 0$.

In the spirit of the grand strategy sketched above, we should now look for a feasible solution that yields a higher value of z. Finding such a solution is not difficult. For example, if we keep $x_2 = x_3 = 0$ and increase the value of x_1, we obtain $z = 5x_1 > 0$. Thus, if we keep $x_2 = x_3 = 0$ and set $x_1 = 1$, we obtain $z = 5$ (and $x_4 = 3$, $x_5 = 7$, $x_6 = 5$). Better yet, if we keep $x_2 = x_3 = 0$ and set $x_1 = 2$, we obtain $z = 10$ (and $x_4 = 1$, $x_5 = 3$, $x_6 = 2$). However, if we keep $x_2 = x_3 = 0$ and set $x_1 = 3$, we obtain $z = 15$ and $x_4 = x_5 = x_6 = -1$; this won't do, since feasibility requires $x_i \geq 0$ for every i. The moral is that we cannot increase x_1 too much. The question is: *Just how much can we increase x_1 (keeping $x_2 = x_3 = 0$ at the same time) and still maintain feasibility* ($x_4, x_5, x_6 \geq 0$)?

The condition $x_4 = 5 - 2x_1 - 3x_2 - x_3 \geq 0$ implies $x_1 \leq \frac{5}{2}$; similarly, $x_5 \geq 0$ implies $x_1 \leq \frac{11}{4}$ and $x_6 \geq 0$ implies $x_1 \leq \frac{8}{3}$. Of these three bounds, the first is the most stringent. Increasing x_1 up to that bound we obtain our next solution,

$$x_1 = \frac{5}{2}, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 1, \quad x_6 = \frac{1}{2}. \tag{2.6}$$

Note that this solution yields $z = \frac{25}{2}$, which is indeed an improvement over $z = 0$.

Next, we should look for a feasible solution that is even better than (2.6). However, this task seems a little more difficult. What made the first iteration so easy? We had at our disposal not only the feasible solution (2.5), but also the system of linear equations (2.3), which guided us in our quest for an improved feasible solution. If we wish to continue in a similar way, we should manufacture a new system of linear equations that relates to (2.6) much as system (2.3) relates to (2.5).

What properties should the new system have? Note that (2.3) expresses the variables that assume positive values in (2.5) in terms of the variables that assume zero values in (2.5). Similarly, the new system should express those variables that assume positive values in (2.6) in terms of the variables that assume zero values in (2.6): in short, it should express x_1, x_5, x_6 (as well as z) in terms of x_2, x_3, and x_4. In particular, the variable x_1, which just changed its value from zero to positive should change its position from the right-hand side to the left-hand side of the system of equations. Similarly, the variable x_4, which just changed its value from positive to zero, should move from the left-hand side to the right-hand side.

To construct the new system, we shall begin with the newcomer to the left-hand side, namely, the variable x_1. The desired formula for x_1 in terms of x_2, x_3, x_4 is obtained easily from the first equation in (2.3):

$$x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4. \tag{2.7}$$

Next, in order to express x_5, x_6, and z in terms of x_2, x_3, x_4, we simply substitute from (2.7) into the corresponding rows of (2.3):

$$\begin{aligned}
x_5 &= 11 - 4\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - x_2 - 2x_3 \\
&= 1 + 5x_2 + 2x_4, \\
x_6 &= 8 - 3\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) - 4x_2 - 2x_3 \\
&= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4, \\
z &= 5\left(\frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4\right) + 4x_2 + 3x_3 \\
&= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4.
\end{aligned}$$

Hence our new system reads

$$\begin{aligned}
x_1 &= \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_5 &= 1 + 5x_2 \qquad\qquad + 2x_4 \\
x_6 &= \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
z &= \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4.
\end{aligned} \tag{2.8}$$

As we did in the first iteration, we shall now try to increase the value of z by increasing the value of a suitably chosen right-hand side variable, while at the same time keeping the remaining right-hand side variables fixed at zero. Note that increases in the values of x_2 or x_4 would bring about *decreases* in the value of z, which is very much against our intentions. Thus, we have no choice: the right-hand side variable to increase its value is necessarily x_3. How much can we increase x_3? The answer can be read directly from system (2.8): with $x_2 = x_4 = 0$, the constraint $x_1 \geq 0$ implies $x_3 \leq 5$, the constraint $x_5 \geq 0$ imposes no restriction at all, and the constraint $x_6 \geq 0$ implies $x_3 \leq 1$. Hence, $x_3 = 1$ is the best we can do; our new solution is

$$x_1 = 2, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 1, \quad x_6 = 0. \tag{2.9}$$

(Note that the value of z just increased from 12.5 to 13.)

As we have learned, getting just the improved solution isn't good enough; we also want a system of linear equations to go with (2.9). In this system, the positive-valued variables x_1, x_3, x_5 will appear on the left, whereas the zero-valued variables x_2, x_4, x_6

will appear on the right. To construct the system, we begin again with the newcomer to the left-hand side, namely, the variable x_3. From the third equation in (2.8), we have $x_3 = 1 + x_2 + 3x_4 - 2x_6$; substituting for x_3 into the remaining equations in (2.8), we obtain

$$
\begin{aligned}
x_3 &= 1 + x_2 + 3x_4 - 2x_6 \\
x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\
x_5 &= 1 + 5x_2 + 2x_4 \\
z &= 13 - 3x_2 - x_4 - x_6.
\end{aligned}
\tag{2.10}
$$

Now it's time for the third iteration. First of all, from the right-hand side of (2.10) we have to choose a variable whose increase brings about an increase of the objective function. However, there is no such variable: indeed, if we increase any of the right-hand side variables x_2, x_4, x_6, we will make the value of z *decrease*. Thus, it seems that we have come to a standstill. In fact, the very presence of this standstill indicates that we are done; we have solved our problem; the solution described by the last table is optimal. Why? The answer lies hidden in the last row of (2.10):

$$z = 13 - 3x_2 - x_4 - x_6. \tag{2.11}$$

Our last solution (2.9) yields $z = 13$; proving that this solution is optimal amounts to proving that every feasible solution satisfies the inequality $z \le 13$. Since every feasible solution $x_1, x_2, \ldots, x_6$ satisfies, among other relations, the inequalities $x_2 \ge 0$, $x_4 \ge 0$, and $x_6 \ge 0$, the desired inequality $z \le 13$ follows directly from (2.11).

DICTIONARIES

In general, given a problem

$$
\begin{aligned}
&\text{maximize} && \sum_{j=1}^{n} c_j x_j \\
&\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \ldots, m) \\
& && x_j \ge 0 \qquad (j = 1, 2, \ldots, n)
\end{aligned}
\tag{2.12}
$$

we first introduce the slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and denote the objective function by z. That is, we define

$$
\begin{aligned}
x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m) \\
z &= \sum_{j=1}^{n} c_j x_j.
\end{aligned}
\tag{2.13}
$$

In the framework of the simplex method, each feasible solution $x_1, x_2, \ldots, x_n$ of (2.12) is represented by $n + m$ *nonnegative* numbers $x_1, x_2, \ldots, x_{n+m}$, with $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ defined by (2.13). In each iteration, the simplex method moves from some feasible solution $x_1, x_2, \ldots, x_{n+m}$ to another feasible solution $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{n+m}$, which is better than the previous one in the sense that

$$\sum_{j=1}^{n} c_j \bar{x}_j > \sum_{j=1}^{n} c_j x_j.$$

(Actually, the last statement is not quite correct: the inequality is not always strict. This point and other subtleties will be discussed in Chapter 3.)

As we have seen, it is convenient to associate a system of linear equations with each of the feasible solutions: such systems make it easier to find the improved feasible solutions. They do so by translating any choice of values of the right-hand side variables into the corresponding values of the left-hand side variables and of the objective function. Following J. E. Strum (1972), we shall refer to these systems as *dictionaries*. Thus, every dictionary associated with (2.12) will be a system of linear equations in the variables $x_1, x_2, \ldots, x_{n+m}$ and z. However, not every system of linear equations in these variables constitutes a dictionary. To begin with, we have defined $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ and z in terms of $x_1, x_2, \ldots, x_n$, and so the $n + m + 1$ variables are heavily interdependent. This interdependence must be captured by every dictionary associated with (2.12): the translations must be correct. More precisely, we shall insist that:

> Every solution of the set of equations comprising a dictionary must be also a solution of (2.13), and vice versa. (2.14)

For example, for every choice of numbers $x_1, x_2, \ldots, x_6$ and z, the following three statements are equivalent:

- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.3),
- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.8),
- $x_1, x_2, \ldots, x_6, z$ constitute a solution of (2.10).

In that sense, the three dictionaries (2.3), (2.8), and (2.10) contain the same information concerning the interdependence among the seven variables. Nevertheless, each of the three dictionaries presents this information in its very own way. The form of (2.3) suggests that we are free to choose the numerical values of x_1, x_2, and x_3 at will, whereupon the values of x_4, x_5, x_6, and z are determined: in this dictionary, the decision variables x_1, x_2, x_3 act as independent variables, while z and the slack variables x_4, x_5, x_6 are dependent on them. Dictionary (2.8) presents x_2, x_3, x_4 as independent and x_1, x_5, x_6, z as dependent. In dictionary (2.10), the independent variables are x_2, x_4, x_6 and the dependent ones are x_3, x_1, x_5, z. In general:

> The equations of every dictionary must express m of the variables $x_1,$

$x_2, \ldots, x_{n+m}$ and the objective function z in terms of the remaining n variables. (2.15)

The properties (2.14) and (2.15) are the defining properties of dictionaries.

In addition to these two properties, dictionaries (2.3), (2.8), and (2.10) have the following property:

> Setting the right-hand side variables at zero and evaluating the left-hand side variables, we arrive at a *feasible* solution.

Dictionaries with this additional property will be called *feasible dictionaries.* Hence, every feasible dictionary describes a feasible solution. However, not every feasible solution is described by a feasible dictionary; for instance, no dictionary describes the feasible solution $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 2, x_5 = 5, x_6 = 3$ of (2.1). Feasible solutions that can be described by dictionaries are called *basic.* The characteristic feature of the simplex method is the fact that it works exclusively with basic feasible solutions and ignores all other feasible solutions.

SECOND EXAMPLE

We shall complete our preview of the simplex method by applying it to another LP problem:

$$
\begin{array}{ll}
\text{maximize} & 5x_1 + 5x_2 + 3x_3 \\
\text{subject to} & x_1 + 3x_2 + x_3 \le 3 \\
& -x_1 + 3x_3 \le 2 \\
& 2x_1 - x_2 + 2x_3 \le 4 \\
& 2x_1 + 3x_2 - x_3 \le 2 \\
& x_1, x_2, x_3 \ge 0.
\end{array}
$$

In this case, the initial feasible dictionary reads

$$
\begin{array}{l}
x_4 = 3 - x_1 - 3x_2 - x_3 \\
x_5 = 2 + x_1 - 3x_3 \\
x_6 = 4 - 2x_1 + x_2 - 2x_3 \\
x_7 = 2 - 2x_1 - 3x_2 + x_3 \\
\hline
z = 5x_1 + 5x_2 + 3x_3.
\end{array}
\tag{2.16}
$$

(Even though the order of the equations in a dictionary is quite irrelevant, we shall make a habit of writing the formula for z last and separating it from the rest of the table by a solid line. Of course, that does *not* mean that the last equation is the sum of the previous ones.) This feasible dictionary describes the feasible solution

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 3, \quad x_5 = 2, \quad x_6 = 4, \quad x_7 = 2.$$

However, there is no need to write this solution down, as we just did: the solution is implicit in the dictionary.

In the first iteration, we shall attempt to increase the value of z by making one of the right-hand side variables positive. At this moment, any of the three variables x_1, x_2, x_3 would do. In small examples, it is common practice to choose the variable that, in the formula for z, has the largest coefficient: the increase in that variable will make z increase at the fastest rate (but not necessarily to the highest level). In our case, this rule leaves us a choice between x_1 and x_2; choosing arbitrarily, we decide to make x_1 positive. As the value of x_1 increases, so does the value of x_5. However, the values of x_4, x_6, and x_7 decrease, and none of them is allowed to become negative. Of the three constraints $x_4 \geq 0$, $x_6 \geq 0$, $x_7 \geq 0$ that impose upper bounds on the increment of x_1, the last constraint $x_7 \geq 0$ is the most stringent: it implies $x_1 \leq 1$. In the improved feasible solution, we shall have $x_1 = 1$ and $x_7 = 0$. Without writing the new solution down, we shall now construct the new dictionary. All we need to know is that x_1 just made its way from the right-hand side to the left, whereas x_7 went in the opposite direction. From the fourth equation in (2.16), we have

$$x_1 = 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7. \tag{2.17}$$

Substituting from (2.17) into the remaining equations of (2.16), we arrive at the desired dictionary

$$\begin{array}{rcl}
x_1 &=& 1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_7\\
x_4 &=& 2 - \frac{3}{2}x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_7\\
x_5 &=& 3 - \frac{3}{2}x_2 - \frac{5}{2}x_3 - \frac{1}{2}x_7\\
x_6 &=& 2 + 4x_2 - 3x_3 + x_7\\
\hline
z &=& 5 - \frac{5}{2}x_2 + \frac{11}{2}x_3 - \frac{5}{2}x_7.
\end{array} \tag{2.18}$$

The construction of (2.18) completes the first iteration of the simplex method.

Digression on Terminology

The variables x_j that appear on the left-hand side of a dictionary are called *basic*; the variables x_j that appear on the right-hand side are *nonbasic*. The basic variables are said to constitute a *basis*. Of course, the basis changes with each iteration: for example, in the first iteration, x_1 entered the basis whereas x_7 left it. In each iteration, we first choose the nonbasic variable that is to enter the basis and then we find out which basic variable must leave the basis. The choice of the *entering* variable is motivated by our desire to increase the value of z; the determination of the *leaving* variable is based on the requirement that all variables must assume nonnegative values. The leaving variable is that basic variable whose nonnegativity imposes the most stringent upper bound on the increment of the entering variable. The formula for the leaving variable appears in the *pivot row* of the dictionary; the computational process of constructing the new dictionary is referred to as *pivoting*.

Back to the Second Example

In our example, the variable to enter the basis during the second iteration is quite unequivocally x_3. This is the only nonbasic variable in (2.18) whose coefficient in the last row is positive. Of the four basic variables, x_6 imposes the most stringent upper bound on the increase of x_3, and, therefore, has to leave the basis. Pivoting, we arrive at our third dictionary,

$$\begin{array}{rcl}
x_3 & = & \frac{2}{3} + \frac{4}{3}x_2 + \frac{1}{3}x_7 - \frac{1}{3}x_6 \\
x_1 & = & \frac{4}{3} - \frac{5}{6}x_2 - \frac{1}{3}x_7 - \frac{1}{6}x_6 \\
x_4 & = & 1 - \frac{7}{2}x_2 \qquad\quad + \frac{1}{2}x_6 \\
x_5 & = & \frac{4}{3} - \frac{29}{6}x_2 - \frac{4}{3}x_7 + \frac{5}{6}x_6 \\
\hline
z & = & \frac{26}{3} + \frac{29}{6}x_2 - \frac{2}{3}x_7 - \frac{11}{6}x_6.
\end{array} \tag{2.19}$$

In the third iteration, the entering variable is x_2 and the leaving variable is x_5. Pivoting yields the dictionary

$$\begin{array}{rcl}
x_2 & = & \frac{8}{29} - \frac{8}{29}x_7 + \frac{5}{29}x_6 - \frac{6}{29}x_5 \\
x_3 & = & \frac{30}{29} - \frac{1}{29}x_7 - \frac{3}{29}x_6 - \frac{8}{29}x_5 \\
x_1 & = & \frac{32}{29} - \frac{3}{29}x_7 - \frac{9}{29}x_6 + \frac{5}{29}x_5 \\
x_4 & = & \frac{1}{29} + \frac{28}{29}x_7 - \frac{3}{29}x_6 + \frac{21}{29}x_5 \\
\hline
z & = & 10 - 2x_7 - x_6 - x_5.
\end{array} \tag{2.20}$$

At this point, no nonbasic variable can enter the basis without making the value of z decrease. Hence, the last dictionary describes an optimal solution of our example. That solution is

$$x_1 = \frac{32}{29}, \quad x_2 = \frac{8}{29}, \quad x_3 = \frac{30}{29}$$

and it yields $z = 10$.

FURTHER REMARKS

The reader may have noticed that, having first carefully laid down the definition of a dictionary, we then proceeded to refer to (2.18), (2.19), and (2.20) as dictionaries, without bothering to verify that they do indeed have property (2.14). Such carelessness can be easily justified. Take, for example, system (2.18). Since (2.18) arises from (2.16) by arithmetical operations (namely, pivoting with x_1 entering and x_7 leaving), every solution of (2.16) must be also a solution of (2.18). The converse is also true, since (2.16) can be obtained from (2.18) by pivoting with x_7 entering and x_1 leaving. Hence, *every solution of* (2.18) *is a solution of* (2.16), *and vice versa*. Similar arguments show that *every solution of* (2.19) *is a solution of* (2.18), *and vice versa*; and that *every solution of* (2.20) *is a solution of* (2.19), *and vice versa*.

Another point of concern is the question of the *uniqueness*, as opposed to the *existence*, of optimal solutions. This question will be of no great interest to us; nevertheless, it is easy to deal with and so we will get it out of the way now. Note that in each of our two examples, we not only found an optimal solution, but we also collected the evidence to prove that there is only one optimal solution. For instance, the final dictionary for our first problem reads

$$\begin{array}{rcl} x_3 &=& 1 + x_2 + 3x_4 - 2x_6 \\ x_1 &=& 2 - 2x_2 - 2x_4 + x_6 \\ x_5 &=& 1 + 5x_2 + 2x_4 \\ \hline z &=& 13 - 3x_2 - x_4 - x_6. \end{array}$$

The last row shows that every feasible solution with $z = 13$ satisfies $x_2 = x_4 = x_6 = 0$; the rest of the dictionary shows that every such solution satisfies $x_3 = 1$, $x_1 = 2$, $x_5 = 1$; therefore, there is just one optimal solution. A similar argument applies to the second problem.

Of course, there are LP problems with more than just one optimal solution; having solved such problems by the simplex method, we can effectively describe all the optimal solutions. For example, consider the following dictionary:

$$\begin{array}{rcl} x_4 &=& 3 + x_2 - 2x_5 + 7x_3 \\ x_1 &=& 1 - 5x_2 + 6x_5 - 8x_3 \\ x_6 &=& 4 + 9x_2 + 2x_5 - x_3 \\ \hline z &=& 8 - x_3. \end{array}$$

The last row shows that every optimal solution satisfies $x_3 = 0$ (but not necessarily $x_2 = 0$ or $x_5 = 0$). For such solutions, the rest of the dictionary implies

$$\begin{aligned} x_4 &= 3 + x_2 - 2x_5 \\ x_1 &= 1 - 5x_2 + 6x_5 \\ x_6 &= 4 + 9x_2 + 2x_5. \end{aligned} \tag{2.21}$$

We conclude that every optimal solution arises by the substitution formulas (2.21) from some x_2 and x_5 such that

$$\begin{aligned} -x_2 + 2x_5 &\le 3 \\ 5x_2 - 6x_5 &\le 1 \\ -9x_2 - 2x_5 &\le 4 \\ x_2, x_5 &\ge 0. \end{aligned}$$

(In fact, the inequality $-9x_2 - 2x_5 \le 4$ is clearly redundant; its validity is forced by $x_2 \ge 0$ and $x_5 \ge 0$.)

There are a few other rough spots we deliberately failed to point out in our overview of the simplex method. We shall discuss them in Chapter 3.

TABLEAU FORMAT

The simplex method is often introduced in a format differing from ours. To outline the more popular *tableau format*, we shall return to the first example of this chapter. To begin, let us write down the equations of the first dictionary in a slightly modified form:

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 + x_4 &=& 5 \\ 4x_1 + x_2 + 2x_3 + x_5 &=& 11 \\ 3x_1 + 4x_2 + 2x_3 + x_6 &=& 8 \\ \hline -z + 5x_1 + 4x_2 + 3x_3 &=& 0. \end{array}$$

Recording just the coefficients at the x_i's, together with the right-hand sides, we obtain our first *tableau*:

$$\begin{array}{cccccccc} 2 & 3 & 1 & 1 & 0 & 0 & & 5 \\ 4 & 1 & 2 & 0 & 1 & 0 & & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & & 8 \\ \hline 5 & 4 & 3 & 0 & 0 & 0 & & 0. \end{array}$$

In a similar way, the equations of the second dictionary,

$$
\begin{array}{rcl}
x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 & = & \frac{5}{2} \\
- 5x_2 \qquad - 2x_4 + x_5 & = & 1 \\
- \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{3}{2}x_4 \qquad + x_6 & = & \frac{1}{2} \\
\hline
-z \quad - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 & = & -\frac{25}{2}
\end{array}
$$

give rise to a second tableau:

$$
\begin{array}{cccccc|c}
1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\
0 & -5 & 0 & -2 & 1 & 0 & 1 \\
0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \\
\hline
0 & -\frac{7}{2} & \frac{1}{2} & -\frac{5}{2} & 0 & 0 & -\frac{25}{2}.
\end{array}
$$

It is a routine matter to translate the pivoting rules, previously derived in terms of dictionaries, into the language of tableaus. The following steps describe the procedure; the reader should have no trouble verifying its correctness. (At any rate, the procedure is not important for our exposition since we do not use the tableau format.)

Step 1. Examine all numbers in the last row (except the one farthest right, which equals the current value of $-z$). If all of them are negative or zero, stop: the tableau describes an optimal solution. Otherwise find the largest of these numbers; the column in which it appears is called the *pivot column* and corresponds to the entering variable.

For example, the pivot column in our first tableau is the first one:

$$
\begin{array}{|c|ccccc|c}
2 & 3 & 1 & 1 & 0 & 0 & 5 \\
4 & 1 & 2 & 0 & 1 & 0 & 11 \\
3 & 4 & 2 & 0 & 0 & 1 & 8 \\
\hline
5 & 4 & 3 & 0 & 0 & 0 & 0
\end{array}
$$

Step 2. For each row whose entry r in the pivot column is positive, look up the entry s in the rightmost column. The row with the smallest ratio $\frac{s}{r}$ is called the *pivot row* and corresponds to the leaving variable. (If all the entries in the pivot column are negative or zero, then the problem is unbounded; more on that in Chapter 3.)

In our example, the pivot row is the first row $\left(\text{with } \frac{s}{r} = \frac{5}{2}\right)$:

2	3	1	1	0	0	5
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0.

Step 3. Divide every entry in the pivot row by the *pivot number*, found in the intersection of the pivot row with the pivot column:

1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{5}{2}$
4	1	2	0	1	0	11
3	4	2	0	0	1	8
5	4	3	0	0	0	0.

Step 4. From every remaining row, subtract a suitable multiple of the new pivot row. This operation is designed to make every entry in the pivot column (except for the pivot number) become zero; hence, the "suitable multiple" results when the new pivot row is multiplied by the entry appearing in the pivot column and in the row in question. (In our example, step 4 results in the second tableau.)

A tableau is nothing but a cryptic recording of a dictionary with all the variables collected on the left-hand side and the symbols for these variables omitted. We shall continue to use dictionaries instead, since they are more explicit. (Of course, nothing prevents the reader tired of writing the same symbols $x_1, x_2, \ldots$ over and over again from using the tableau shorthand.) □

A WARNING

There is often more than one way of describing a particular algorithm; descriptions aimed at clarifying underlying concepts are often quite different from those that suggest efficient computer implementations. The simplex method is no exception. Dictionaries may provide a convenient tool for explaining its basic principles. However, in implementing the method for computer solutions of large problems, considerations of computational efficiency and numerical accuracy overshadow such didactic niceties. We shall begin to study efficient implementations of the simplex method in Chapters 7 and 8.

PROBLEMS

△ **2.1** Solve the following problems by the simplex method:

a.
$$\begin{array}{ll}\text{maximize} & 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} & \begin{aligned} x_1 + x_2 + 2x_3 &\le 4 \\ 2x_1 \phantom{{}+ x_2} + 3x_3 &\le 5 \\ 2x_1 + x_2 + 3x_3 &\le 7 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}\end{array}$$

b.
$$\begin{array}{ll}\text{maximize} & 5x_1 + 6x_2 + 9x_3 + 8x_4 \\ \text{subject to} & \begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\le 5 \\ x_1 + x_2 + 2x_3 + 3x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}\end{array}$$

c.
$$\begin{array}{ll}\text{maximize} & 2x_1 + x_2 \\ \text{subject to} & \begin{aligned} 2x_1 + 3x_2 &\le 3 \\ x_1 + 5x_2 &\le 1 \\ 2x_1 + x_2 &\le 4 \\ 4x_1 + x_2 &\le 5 \\ x_1, x_2 &\ge 0. \end{aligned}\end{array}$$

2.2 Use the simplex method to describe *all* the optimal solutions of the following problem:

$$\begin{array}{ll}\text{maximize} & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ \text{subject to} & \begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\le 5 \\ x_1 + x_2 + 2x_3 + 3x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}\end{array}$$