1

Introduction

In this short chapter, we shall explain what is meant by linear programming and sketch a history of this subject.

A DIET PROBLEM

Polly wonders how much money she must spend on food in order to get all the energy (2,000 kcal), protein (55 g), and calcium (800 mg) that she needs every day. (For iron and vitamins, she will depend on pills. Nutritionists would disapprove, but the introductory example ought to be simple.) She chooses six foods that seem to be cheap sources of the nutrients; her data are collected in Table 1.1.

Table 1.1 Nutritive Value per Serving

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Then she begins to think about her menu. For example, 10 servings of pork with beans would take care of all her needs for only (?) \$1.90 per day. On the other hand, 10 servings of pork with beans is a lot of pork with beans—she would not be able to stomach more than 2 servings a day. She decides to impose servings-per-day limits on all six foods:

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day.

Now, another look at the data shows Polly that 8 servings of milk and 2 servings of cherry pie every day will satisfy the requirements nicely and at a cost of only \$1.12. In fact, she could cut down a little on the pie or the milk or perhaps try a different combination. But so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here. To be systematic, we may speculate about some as yet unspecified menu consisting of x_1 servings of oatmeal, x_2 servings of chicken, x_3 servings of eggs, and so on. In order to stay below the upper limits, that menu must satisfy

$$\begin{aligned}
0 &\le x_1 \le 4 \\
0 &\le x_2 \le 3 \\
0 &\le x_3 \le 2 \\
0 &\le x_4 \le 8 \\
0 &\le x_5 \le 2 \\
0 &\le x_6 \le 2.
\end{aligned} \tag{1.1}$$

And, of course, there are the requirements for energy, protein, and calcium; they lead to the inequalities

$$\begin{aligned}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\ge 2{,}000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\ge 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\ge 800.
\end{aligned} \tag{1.2}$$

If some numbers $x_1, x_2, \ldots, x_6$ satisfy inequalities (1.1) and (1.2), then they describe a satisfactory menu; such a menu will cost, in cents per day,

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6. \tag{1.3}$$

In designing the most economical menu, Polly wants to find numbers $x_1, x_2, \ldots, x_6$ that satisfy (1.1) and (1.2), and make (1.3) as small as possible. As a mathematician would put it, she wants to

$$\begin{array}{ll} \text{minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ \text{subject to} & 0 \le x_1 \le 4 \\ & 0 \le x_2 \le 3 \\ & 0 \le x_3 \le 2 \\ & 0 \le x_4 \le 8 \\ & 0 \le x_5 \le 2 \\ & 0 \le x_6 \le 2 \\ \multicolumn{2}{l}{110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000} \\ \multicolumn{2}{l}{\quad 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55} \\ \multicolumn{2}{l}{\quad 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800.} \end{array} \tag{1.4}$$

Her problem is known as a *diet problem.*

LINEAR PROGRAMMING

Problems of this kind are called "linear programming problems," or "LP problems" for short; linear programming is the branch of applied mathematics concerned with these problems. Here are other examples:

$$\begin{array}{ll} \text{maximize} & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 \le 5 \\ & 4x_1 + x_2 + 2x_3 \le 11 \\ & 3x_1 + 4x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{array} \tag{1.5}$$

(with "$x_1, x_2, x_3 \ge 0$" used as shorthand for "$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$") or

$$\begin{array}{ll} \text{minimize} & 3x_1 - x_2 \\ \text{subject to} & -x_1 + 6x_2 - x_3 + x_4 \ge -3 \\ & 7x_2 + 2x_4 = 5 \\ & x_1 + x_2 + x_3 = 1 \\ & x_3 + x_4 \le 2 \\ & x_2, x_3 \ge 0. \end{array} \tag{1.6}$$

In general, if $c_1, c_2, \ldots, c_n$ are real numbers, then the function f of real variables $x_1, x_2, \ldots, x_n$ defined by

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{j=1}^{n} c_jx_j$$

is called a *linear function.* If f is a linear function and if b is a real number, then the equation

$$f(x_1, x_2, \ldots, x_n) = b$$

is called a *linear equation* and the inequalities

$$f(x_1, x_2, \ldots, x_n) \le b$$
$$f(x_1, x_2, \ldots, x_n) \ge b$$

are called *linear inequalities.* Linear equations and linear inequalities are both referred to as *linear constraints.* Finally, a *linear programming problem* is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints. We shall usually attach different subscripts i to different constraints and different subscripts j to different variables. For simplicity of exposition, we shall restrict ourselves in Chapters 1–7 to LP problems of the following form:

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, 2, \ldots, m) \\ & && x_j \ge 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{1.7}$$

These problems will be referred to as LP problems in the *standard form.* (The reader should be warned that the terminology is far from unified; several authors prefer the terms *canonical* or *symmetric* form, and others reserve these adjectives for altogether different problems.) For example, (1.5) is a problem in the standard form (with $n = 3, m = 3, a_{11} = 2, a_{12} = 3$, and so on). What distinguishes the problems in the standard form from the rest? First, all of their constraints are linear *inequalities.* Secondly, the last n of the $m + n$ constraints in (1.7) are very special: they simply stipulate that none of the n variables may assume negative values. Such constraints are called *nonnegativity constraints.* (Note that problem (1.6) differs from the standard form on both counts: two of its constraints are linear equations and the variables x_1, x_4 may assume negative values.)

The linear function that is to be maximized or minimized in an LP problem is called the *objective function* of that problem. For example, the function z of variables $x_1, x_2, x_3, x_4, x_5, x_6$ defined by

$$z(x_1, x_2, \ldots, x_6) = 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

is the objective function of Polly's diet problem (1.4). Numbers $x_1, x_2, \ldots, x_n$ that satisfy all the constraints of an LP problem are said to constitute a *feasible solution* of that problem. For instance, we have observed that

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 8, \quad x_5 = 2, \quad x_6 = 0$$

is a feasible solution of (1.4). Finally, a feasible solution that maximizes the objective function (or minimizes it, depending on the form of the problem) is called an *optimal solution*; the corresponding value of the objective function is called the *optimal value* of the problem. As it turns out, the unique optimal solution of (1.4) is

$$x_1 = 4, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 4.5, \quad x_5 = 2, \quad x_6 = 0$$

or simply (4, 0, 0, 4.5, 2, 0). Accordingly, the optimal value of (1.4) is 92.5. Not every LP problem has a unique optimal solution; some problems have many different optimal solutions and others have no optimal solutions at all. The latter may occur for one of two radically different reasons: either there are no feasible solutions at all or there are, in a sense, too many of them. The first case may be illustrated on the problem

$$\begin{aligned} \text{maximize} \quad & 3x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & -2x_1 - 2x_2 \le -10 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{1.8}$$

which has no feasible solutions at all. Such problems are called *infeasible*. On the other hand, even though the problem

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \le -1 \\ & -x_1 - 2x_2 \le -2 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{1.9}$$

does have feasible solutions, none of them is optimal: for every number M there is a feasible solution x_1, x_2 such that $x_1 - x_2 > M$. In a sense, (1.9) has such an abundance of feasible solutions that none of them can aspire to be the best. Problems with this property are called *unbounded*. As we shall prove later (Theorem 3.4), every linear programming problem belongs to one of the three categories noted here: it has an optimal solution, is infeasible, or is unbounded.

HISTORY OF LINEAR PROGRAMMING

As mathematical disciplines go, linear programming is quite young. It started in 1947 when G. B. Dantzig designed the "simplex method" for solving linear programming formulations of U.S. Air Force planning problems. What followed was an exciting period of rapid development in this new field. It soon became clear that a surprisingly wide range of apparently unrelated problems in production management could be stated in linear programming terms and, most importantly, solved by the simplex method. Such problems, if noticed at all, had traditionally been tackled by a hit-or-miss approach guided only by experience and intuition. The use of linear programming often brought about a considerable increase in the efficiency of the whole operation. (Until then, expansion of the efficiency frontier usually came from technological innovations. This new way to increase efficiency—*under existing technological conditions*—by improvements in organization and planning, made many managers appreciate the practical importance of mathematics. At least, it made them aware of the advantage of stating their decision problems in clear-cut and well-defined terms.) As the popularity of linear programming theory increased, applications in new areas occurred, many of them far from obvious. In turn, these applications stimulated further theoretical research by pointing out the need for solving problems that would have otherwise seemed uninteresting. In this fascinating interplay between theory and applications, a new branch of applied mathematics established itself.

As calculus developed from the seventeenth century's need to solve problems of mechanics, linear programming developed from the twentieth century's need to solve problems of management. Yet other profound influences stimulated the evolution of the new field from its very inception. Economics was one of them: as early as 1947, T. C. Koopmans began pointing out that linear programming provided an excellent framework for the analysis of classical economic theories, such as the renowned system proposed in 1874 by L. Walras. On the other hand, linear programming brought together previously known theorems of pure mathematics concerning such diverse topics as the geometry of convex sets, extremal problems of combinatorial nature, and the theory of two-person games. Finally, it was fortunate and perhaps even inevitable that linear programming developed concurrently with modern computer technology: without electronic computers, present-day large-scale linear programming would be unthinkable.

Scientific fields are rarely born overnight; with the advantage of hindsight, one can often track down the sources that paved the way for the decisive breakthrough. The field of linear programming is no exception. At the core of its mathematical theory is the study of systems of linear inequalities; such systems were investigated by Fourier as far back as 1826. Since then, quite a few other mathematicians have considered the subject, although none of them has devised an algorithm whose efficiency has come close to that of the simplex method. Nevertheless, some of them proved various special cases of a fundamental theorem that is now called the *duality theorem* of linear programming. On the applied side, L. V. Kantorovich pointed out the practical significance of a restricted class of LP problems, and proposed a rudimentary algorithm for their solution as early as 1939. Regrettably, this effort remained neglected in the U.S.S.R. and unknown elsewhere until long after linear programming became an elegant theory through the independent work of Dantzig and others.

In the 1970s, linear programming came twice to public attention. On October 14, 1975, the Royal Sweden Academy of Sciences awarded the Nobel Prize in economic science to L. V. Kantorovich and T. C. Koopmans "for their contributions to the theory of optimum allocation of resources." (As the reader may know, there is no Nobel Prize in mathematics. Apparently the Academy regarded the work of G. B. Dantzig, who is universally recognized as the father of linear programming, as being too mathematical.) The second event was even more dramatic. Ever since the invention of the simplex method, mathematicians had been looking for a *theoretically* satisfactory algorithm to solve LP problems. (A word of explanation is in order: theoretical criteria for judging the efficiency of algorithms are quite different from practical ones. Thus, an algorithm like the simplex method, which is eminently satisfactory in practical applications, may be found theoretically unsatisfactory. The converse is also true: theoretically satisfactory algorithms may be thoroughly useless in practice. We shall return to this distinction in

Chapter 4.) The breakthrough came in 1979 when L. G. Khachian published a description of such an algorithm (based on earlier works by Shor, and by Judin and Nemirovskii). Newspapers around the world published reports of this result, some of them full of hilarious misinterpretations. We shall present the algorithm in the appendix.

For a thorough survey of the history of linear programming, the reader is referred to Chapter 2 of Dantzig's monograph (1963). References to many applications of linear programming may be found in Riley and Gass (1958). Some of the more recent applications are referenced in Gass (1975). □

PROBLEMS

Answers to problems marked with the symbol △ are found at the back of the book.

1.1 Which of the problems below are in the standard form?

a.
$$\begin{array}{ll} \text{Maximize} & 3x_1 - 5x_2 \\ \text{subject to} & 4x_1 + 5x_2 \geq 3 \\ & 6x_1 - 6x_2 = 7 \\ & x_1 + 8x_2 \leq 20 \\ & x_1, x_2 \geq 0. \end{array}$$

b.
$$\begin{array}{ll} \text{Minimize} & 3x_1 + x_2 + 4x_3 + x_4 + 5x_5 \\ \text{subject to} & 9x_1 + 2x_2 + 6x_3 + 5x_4 + 3x_5 \leq 5 \\ & 8x_1 + 9x_2 + 7x_3 + 9x_4 + 3x_5 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

c.
$$\begin{array}{ll} \text{Maximize} & 8x_1 - 4x_2 \\ \text{subject to} & 3x_1 + x_2 \leq 7 \\ & 9x_1 + 5x_2 \leq -2 \\ & x_1, x_2 \geq 0. \end{array}$$

1.2 State in the standard form:

$$\begin{array}{ll} \text{minimize} & -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5 \\ \text{subject to} & 6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 \geq 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

1.3 Prove that (1.8) is infeasible and (1.9) is unbounded.

△**1.4** Find necessary and sufficient conditions for the numbers s and t to make the LP problem

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & sx_1 + tx_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a. have an optimal solution,

b. be infeasible,

c. be unbounded.

△ **1.5** Prove or disprove: If problem (1.7) is unbounded, then there is a subscript k such that the problem

$$\begin{array}{lll} \text{maximize} & x_k & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij}x_j \le b_i & (i = 1, 2, \ldots, m) \\ & x_j \ge 0 & (j = 1, 2, \ldots, n) \end{array}$$

is unbounded.

△ **1.6** [Adapted from Greene et al. (1959).] A meat packing plant produces 480 hams, 400 pork bellies, and 230 picnic hams every day; each of these products can be sold either fresh or smoked. The total number of hams, bellies, and picnics that can be smoked during a normal working day is 420; in addition, up to 250 products can be smoked on overtime at a higher cost. The *net* profits are as follows:

	Fresh	Smoked on regular time	Smoked on overtime
Hams	\$8	\$14	\$11
Bellies	\$4	\$12	\$7
Picnics	\$4	\$13	\$9

For example, the following schedule yields a total net profit of \$9,965:

	Fresh	Smoked	Smoked (overtime)
Hams	165	280	35
Bellies	295	70	35
Picnics	55	70	105

The objective is to find the schedule that maximizes the total net profit. Formulate as an LP problem in the standard form.

1.7 [Adapted from Charnes et al. (1952).] An oil refinery produces four types of raw gasoline: alkylate, catalytic-cracked, straight-run, and isopentane. Two important characteristics of each gasoline are its performance number PN (indicating antiknock properties) and its vapor pressure RVP (indicating volatility). These two characteristics, together with the production levels in barrels per day, are as follows:

	PN	RVP	Barrels produced
Alkylate	107	5	3,814
Catalytic-cracked	93	8	2,666
Straight-run	87	4	4,016
Isopentane	108	21	1,300

These gasolines can be sold either raw, at \$4.83 per barrel, or blended into aviation gasolines (Avgas A and/or Avgas B). Quality standards impose certain requirements on the aviation gasolines; these requirements, together with the selling prices, are as follows:

	PN	RVP	Price per barrel
Avgas A	at least 100	at most 7	\$6.45
Avgas B	at least 91	at most 7	\$5.91

The PN and RVP of each mixture are simply weighted averages of the PNs and RVPs of its constituents. For example, the refinery could adopt the following strategy:

- Blend 2,666 barrels of alkylate and 2,666 barrels of catalytic into 5,332 barrels of Avgas A with

$$\mathrm{PN} = \frac{(2{,}666 \times 107) + (2{,}666 \times 93)}{5{,}332} = 100$$

$$\mathrm{RVP} = \frac{(2{,}666 \times 5) + (2{,}666 \times 8)}{5{,}332} = 6.5.$$

- Blend 1,148 barrels of alkylate, 4,016 barrels of straight-run, and 1,024 barrels of isopentane into 6,188 barrels of Avgas B with

$$\mathrm{PN} = \frac{(1{,}148 \times 107) + (4{,}016 \times 87) + (1{,}024 \times 108)}{6{,}188} \doteq 94.2$$

$$\mathrm{RVP} = \frac{(1{,}148 \times 5) + (4{,}016 \times 4) + (1{,}024 \times 21)}{6{,}188} \doteq 7.$$

  Sell 276 barrels of isopentane raw.

This sample plan yields a total profit of

$$(5{,}332 \times 6.45) + (6{,}188 \times 5.91) + (276 \times 4.83) \doteq \$72{,}296.$$

The refinery aims for the plan that yields the largest possible profit. Formulate as an LP problem in the standard form.

1.8 An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay \$20 for each radio delivered by the end of the first week, \$18 for those delivered by the end of the second week, \$16 by the end of the third week, and \$14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll till the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are \$200; the weekly wages of a trainee are \$100. The production costs, excluding the worker's wages, are \$5 per radio.

For example, the company could adopt the following program.

First week: 10 assemblers, 30 instructors, 90 trainees
Workers' wages: \$8,000

Trainees' wages: \$9,000
Profit from 500 radios: \$7,500
Net loss: \$9,500

Second week: 120 assemblers, 10 instructors, 30 trainees
Workers' wages: \$26,000
Trainees' wages: \$3,000
Profit from 6,000 radios: \$78,000
Net profit: \$49,000

Third week: 160 assemblers
Workers' wages: \$32,000
Profit from 8,000 radios: \$88,000
Net profit: \$56,000

Fourth week: 110 assemblers, 50 idle
Workers' wages: \$32,000
Profit from 5,500 radios: \$49,500
Net profit: \$17,500

This program, leading to a total net profit of \$113,000, is one of many possible programs. The company's aim is to maximize the total net profit. Formulate as an LP problem (not necessarily in the standard form).

△ **1.9** [S. Masuda (1970); see also V. Chvátal (1983).] The *bicycle problem* involves n people who have to travel a distance of ten miles, and have one single-seat bicycle at their disposal. The data are specified by the walking speed w_j and the bicycling speed b_j of each person j $(j = 1, 2, \ldots, n)$; the task is to minimize the arrival time of the last person. (Can you solve the case of $n = 3$ and $w_1 = 4$, $w_2 = w_3 = 2$, $b_1 = 16$, $b_2 = b_3 = 12$?) Show that the optimal value of the LP problem

$$
\begin{array}{lrll}
\text{minimize} & t & & \\
\text{subject to} & t - x_j - x'_j - y_j - y'_j \geq & 0 & (j = 1, 2, \ldots, n) \\
& t \quad - \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} y'_j \geq & 0 & \\
& w_j x_j - w_j x'_j + b_j y_j - b_j y'_j = & 10 & (j = 1, 2, \ldots, n) \\
& \sum_{j=1}^{n} b_j y_j - \sum_{j=1}^{n} b_j y'_j \leq & 10 & \\
& x_j, x'_j, y_j, y'_j \geq & 0 & (j = 1, 2, \ldots, n)
\end{array}
$$

provides a lower bound on the optimal value of the bicycle problem.