Note

Container ship stowage problem: complexity and connection to the coloring of circle graphs

Mordecai Avriel, Michal Penn*, Naomi Shpirer

Faculty of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa 3200, Israel

Abstract

This paper deals with a stowage plan for containers in a container ship. Since the approach to the containers on board the ship is only from above, it is often the case that containers have to be shifted. Shifting is defined as the temporary removal from and placement back of containers onto a stack of containers. Our aim is to find a stowage plan that minimizes the shifting cost. We show that the shift problem is NP-complete. We also show a relation between the stowage problem and the coloring of circle graphs problem. Using this relation we slightly improve Unger's upper bound on the coloring number of circle graphs. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

This paper was motivated by the operation of a container ship that calls many ports, and in each port it loads and unloads containers. Since, the approach to the containers on board the ship is only from above, it is often the case that containers have to be shifted. Shifting is defined as the temporary removal from and placement back of containers onto a stack of containers. The need for shiftings arises, for example, in a vertical stack of containers if there is a container placed inside the stack that has port j as its destination, while the containers on top of it have destinations further away from port j. In this case the latter containers have to be shifted. The cost of shiftings for a large ship can be very considerable.

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* Corresponding author. Fax: 972 4 8235194.

E-mail address: mpenn@ie.technion.ac.il (M. Penn).

The number of containers shifted along the ship's route is greatly affected by planning the placement of containers on board the ship. The task of determining the best container placement is called stowage planning, see e.g. [16,17]. Whereas most stowage plans are based on port efficiency, and stability–strength considerations of the ship, not much attention has been given to devise a plan that minimize the number of shifts. Some attempts in this direction can be found in [2–4]. A related problem that can be modelled similarly is the problem of dispatching trams on storage yard (see [5]).

In this paper we address the computational complexity of this optimization problem. Aslidis [1] presents a polynomial-time algorithm for solving the *single* column case. We, further, show that the general optimization problem is NP-complete. It is clear that if the number of columns in a rectangular bay is very large, we can devise a plan that no shiftings would be necessary along the ship's route. We derive upper and lower bounds on the number of columns for which a plan can be found in polynomial time that will result in zero shifts. For special types of transportation (source–destination) matrices we derive the exact number of such columns. Further, we show that finding the minimum number of columns for which there is a zero shifts stowage plan is equivalent to finding the coloring number of circle graphs. Using this relation we slightly improve Unger's upper bound on the coloring number of circle graphs [18]. This bound and the appropriate coloring can be computed in polynomial time by using our previous results and Unger's approximation algorithm.

Consider a container ship consisting of a single bay for container stowage. The bay has C vertical columns and R rows. If each column in the bay has a finite number of rows, then the bay is referred to as a *capacitated bay*, otherwise it is referred to as an *uncapacitated bay*. Assume, for simplicity, that all the containers are of the same standard size. The ship starts its service route at port 1 with its bay empty of containers, and it sequentially visits ports $2, 3, \ldots, n$. In each port $i = 1, \ldots, n - 1$, containers can be loaded to destinations $i + 1, \ldots, n$. In the last port n, the ship is emptied of all its containers. The placement of containers in a bay when the ship leaves port i remains unchanged until arrival at port $i + 1$. Let $T = [T_{ij}]$ be the $(n-1) \times (n-1)$ *transportation matrix*, where T_{ij} is the number of containers originating at port i with destination j, $j = i + 1, \ldots, n$. Thus, $T_{ij} = 0$ for all $i \geqslant j$. Note, that the indices of the diagonal of T are $T_{i,i+1}$. Assume that the transportation matrix is known before the ship starts its service route.

A container is a *j-container* if its destination is port j. A *j-container* is an *ij-container* if its origin is i. The set of all *ij-containers* is referred to as an *ij-group*. The problem of finding a stowage plan with the smallest number of shifts, is referred to as the *minimum shift problem*. Consider the following decision *shift problem*: Given a transportation matrix, a nonnegative integer s, and a bay, is there a stowage plan with a cost of at most s shifts? A shift problem with an uncapacitated bay is referred to as an *uncapacitated shift problem*. Consider the following decision *uncapacitated s (zero)-shift problem*: Given a transportation matrix and an uncapacitated bay, is there a stowage plan with a cost of at most s (zero) shifts?

Note that it is easy to see that given a transportation matrix and an uncapacitated bay, there is always an optimal stowage plan in which the containers of each ij-group are assigned to successive slots in one of the columns in the bay. Thus, any solution for the uncapacitated zero-shift problem, can be derived easily from a solution to a related problem where any ij-group of containers is replaced by a single ij-container. However, if the bay is capacitated, then the above remark does not necessarily hold.

2. Zero-shifts and coloring of overlap graphs

We show here the connection between the zero-shift problem and coloring of overlap graphs. We start with some definitions. An $[i,j]$-interval is $\{x \in R: i \leqslant x \leqslant j\}$, where R is the set of the real numbers. To each ij-group of containers with $T_{ij} > 0$, assign an $[i,j]$-interval on the line. Thus, for every transportation matrix T, there is a unique set of intervals. Observe as well that for every set of $[i,j]$-intervals with i,j nonnegative integers we can associate a corresponding transportation matrix T.

We say that two intervals, say an $[i,j]$-interval and a $[k,l]$-interval, *overlap* if $i < k < j < l$. Observe that two intervals overlap if, and only if, their corresponding elements in the matrix, say T_{ij} and T_{kl} with $i < k < j < l$, do not satisfy the following condition: If $T_{ij} > 0$ then $T_{kl} = 0$. Thus, in order to avoid shifts, the ij-containers and the kl-containers must be located in different columns. We say that an $[i,j]$-interval *contains* a $[k,l]$-interval if $i < k < l < j$. Observe, further, that if one interval contains the other, say an $[i,j]$-interval contains a $[k,l]$-interval, then if the kl-container is loaded in a slot above the ij-container then the kl-container is not blocking the ij-container at its final destination in port j.

Consider a family $\mathscr{I} = \{I_1, \ldots, I_n\}$ of intervals on a line. The *overlap graph* of $\mathscr{I}$ is defined to be a simple graph in which each vertex corresponds to an interval, and two vertices are joined together if and only if the two corresponding intervals overlap [12]. Clearly, for each transportation matrix we can construct its corresponding overlap graph, and this graph consists of at most $n(n-1)/2$ vertices, where n is the number of ports.

A graph is *C-colorable* if there is a partition of its vertices into C sets, such that no two adjacent vertices are in the same set. The problem of finding χ, the minimum number of colors needed for coloring a given graph, is called the *coloring problem*. The *C-coloring problem* is as follows: Given a graph, is it C-colorable? Using the relations between overlap graphs and transportation matrices we have the following lemma which is easy to prove.

Lemma 2.1. *Given a transportation matrix T and an uncapacitated bay, the corresponding overlap graph is C-colorable if, and only if, C columns are sufficient for obtaining a zero-shifts plan.*

Thus, the uncapacitated zero-shift problem is shown to be equivalent to the *C*-coloring problem of overlap graphs. Note that one can show that, under certain ordering of the intervals corresponding to a transportation matrix, the zero-shift stowage problem is equivalent to the problem of sorting using stacks in parallel. This sorting problem is shown to be equivalent to overlap graph coloring [9].

3. NP-completeness

Clearly, the (uncapacitated) shift problem and the (uncapacitated) zero shift problem are in NP. Observe that the NP-completeness of the uncapacitated zero-shift problem implies the NP-completeness of the uncapacitated shift problem. Based on the above observation, our aim is to prove the NP-completeness of the uncapacitated zero-shift problem.

In 1980, Garey et al. [11] have shown that the *C*-coloring problem of overlap graphs is NP-complete. Denoting by C^* the minimum number of uncapacitated columns needed for a zero-shifts plan to exist, it follows that finding C^* is NP-complete. However, for fixed *C*, the complexity of the *C*-coloring problem remained unknown for several years. In 1988, Unger [18] partially solved the problem by showing that the *C*-coloring problem of overlap graphs is NP-complete for any fixed $C \geqslant 4$. He completed the picture in 1992 [19] by presenting an $O(|V| \log |V|)$ algorithm for 3-coloring of overlap graphs, where *V* denotes the set of the vertices of the overlap graph. Observe that 2-coloring of an overlap graph can be done in polynomial time (recognition of bipartite graphs). Hence,

Theorem 3.1. *Let C be the number of columns in a uncapacitated bay. Then, the uncapacitated shift problem is NP-complete for $C \geqslant 4$.*

Note that the above theorem implies that for any given *s*, the uncapacitated *s*-shift problem is NP-complete for $C \geqslant 4$. Observe also that the uncapacitated shift problem with 2 or 3 uncapacitated columns remains unsolved. For a single uncapacitated column there is an $O(|n|^3)$ time algorithm [1], where *n* indicates the number of ports.

We turn now to discuss the capacitated shift problem. Consider the following decision problem. Given a transportation matrix, a nonnegative integer *s*, and a capacitated bay, is there a stowage plan with a cost of at most *s* shifts? It follows from Theorem 3.1 that if $C \geqslant 4$ then the capacitated shift problem cannot be polynomially solvable for every *R*, the number of rows in each column. However, for fixed *R*, the complexity of the problem is still unknown. Clearly, if $R = 1$ then the capacitated minimum shift problem is polynomially solvable; yet even for $R = 2$, solving the problem is not trivial.

4. Bounds on C^*

In the previous section we have shown that given a transportation matrix, finding C^*, the minimum number of uncapacitated columns needed for a zero-shifts plan to exist, is

NP-complete. Herein, we present some bounds on this number. Based on Lemma 2.1, these bounds are also bounds on the coloring number of the appropriate overlap graph. The following definition is needed in the sequel.

Define a *k-clique* of a graph to be a completely connected subgraph on k vertices (e.g. [8]). Let ω be the cardinality of a maximum clique in the graph. It is shown in [15] that a maximum clique in an overlap graph can be found in $O(|V|^2)$ time. Given an overlap graph, it is shown in [18] that a 2ω-coloring is always possible and can be found in $O(|V|^2)$ time, but the $(2\omega - 1)$-coloring problem is NP-complete.

Let T be a transportation matrix and G its corresponding overlap graph. Let C^* be as defined above, and let ω be the cardinality of a maximum clique in G. Then, 2ω serves as an upper bound on C^* by using the approximation algorithm of Unger [18].

Also, ω is a lower bound on C^*, since χ, the coloring number of a graph, satisfies $\chi \geqslant \omega$. Thus, for any transportation matrix T, $\omega \leqslant C^* \leqslant 2\omega$. We show below a better bound for general transportation matrices and tight bounds for special types of transportation matrices.

Denote by 1_n-*matrix* the $(n-1) \times (n-1)$ upper triangular matrix with all entries on the diagonal and above it equal to 1. Denote by $\lfloor \alpha \rfloor$ the lower integer part of α and by $\lceil \alpha \rceil$ the upper integer part of α.

Lemma 4.1. *Let G be the overlap graph corresponding to a 1_n-matrix. Then a maximum clique in G is of size $\lfloor n/2 \rfloor$.*

Proof. Consider the intervals and the overlap graph associated with the 1_n-matrix. Assume a clique in the overlap graph contains a vertex that corresponds to an interval of size k. Then, the size of that clique is at most k. Hence, if a vertex of the clique corresponds to an interval of size $k \leqslant \lfloor n/2 \rfloor$, then the clique size is at most $\lfloor n/2 \rfloor$. Now, if all the vertices of the clique correspond to intervals of size $> \lfloor n/2 \rfloor$, then there are at most $\lfloor n/2 \rfloor$ such vertices. Therefore $\omega \leqslant \lfloor n/2 \rfloor$. To see the equality, one should consider a clique where each vertex of the clique corresponds to an interval of size $\lfloor n/2 \rfloor$. Since there are at least $\lfloor n/2 \rfloor$ such vertices, the corresponding clique would be of size $\lfloor n/2 \rfloor$. Thus, a maximum clique is of size $\lfloor n/2 \rfloor$, and the proof of the lemma is complete. $\square$

Note that if n is even, then there is a unique maximum clique of size $n/2$.

Lemma 4.2. *Let T be an $(n-1) \times (n-1)$ transportation matrix with no zero entry on or above the diagonal. Then,*

$$\left\lfloor \frac{n}{2} \right\rfloor \leqslant C^* \leqslant \left\lceil \frac{n}{2} \right\rceil.$$

Proof. Clearly, we can assume without loss of generality that T is a 1_n-transportation matrix. Based on Lemma 4.1, the fact that C^* and the coloring number are the same (Lemma 2.1), and that always $\chi \geqslant \omega$, we obtain that $\lfloor n/2 \rfloor \leqslant C^*$.

Color →	1	2	3	$\cdots$	$\frac{n}{2}-1$	$\frac{n}{2}$
	3	5	7	$\cdots$	$n-1$	$n+1$
A_{ij} values	4	6	8	$\cdots$	n	$n+2$
→	$n+3$	$n+5$	$n+7$	$\cdots$	$2n-1$	
	$n+4$	$n+6$	$n+8$	$\cdots$		

Fig. 1. Coloring the vertices of G according to their A_{ij} values.

It will follow from the two lemmas to follow that $C^* \leqslant \lceil n/2 \rceil$. We originally proved this part by presenting a simple polynomial-time algorithm for obtaining a zero-shift plan. However, after having communicated our result to Ron Holzman, he pointed out how to present our zero-shifts plan in a bay with $\lceil n/2 \rceil$ uncapacitated columns, as coloring of the appropriate overlap graph. We have chosen to present here Holtzman's proofs [14]. □

Lemma 4.3. *Let T be a 1_n-transportation matrix and let G be its corresponding overlap graph. Then, G is $\lceil n/2 \rceil$-colorable.*

Proof. For $1 \leqslant i < j \leqslant n$, let the $[i,j]$-interval be the one corresponding to the ij-container. For each $[i,j]$-interval let $A_{ij} = i + j$. Observe that if two intervals, say $[i,j]$ and $[k,l]$, overlap, then

$$2 \leqslant |A_{ij} - A_{kl}| \leqslant n - 2. \tag{1}$$

Below, we show an $\lceil n/2 \rceil$ coloring for any even n. One can obtain in a similar way, an $\lceil n/2 \rceil$ coloring for any odd n. We color the vertices according to their A_{ij} values as shown in Fig. 1. To see that the above coloring is proper, one should observe that (1) does not hold for each set of vertices having the same color. □

Lemma 4.4. *Let T be an $(n-1) \times (n-1)$ transportation matrix with no zero entry on or above the diagonal, let n be an odd number, and let G be its associated overlap graph. Then $\chi > \omega = \lfloor n/2 \rfloor$.*

Proof. Consider the set of all $[i,j]$-intervals, each corresponding to an ij-group of containers. Consider, further, the set V of all vertices of G. For each $[i,j]$-interval we denote by $v_{i,j}$ its corresponding vertex. Recall that n is odd and let $n = 2k + 1$. Observe that by Lemma 4.1 $\omega = k$. Consider the following two maximum cliques. The first one is $\{v_{1,k+1}, v_{2,k+3}, v_{3,k+4}, \ldots, v_{k,2k+1}\}$ and the second one is $\{v_{1,k+2}, v_{2,k+3}, v_{3,k+4}, \ldots, v_{k,2k+1}\}$. Observe that the two cliques differ only by one vertex. Thus, in any proper k-coloring of G, $v_{1,k+1}$ and $v_{1,k+2}$ would have the same color.

Consider now another pair of maximum cliques. One is $\{v_{2,k+2}, v_{3,k+3}, v_{4,k+4}, \ldots, v_{k+1,2k+1}\}$ and the other is $\{v_{1,k+2}, v_{3,k+3}, v_{4,k+4}, \ldots, v_{k+1,2k+1}\}$. These two cliques also differ by one vertex and hence in every proper k-coloring $v_{2,k+2}$ and $v_{1,k+2}$ would have the same color.

Recall that for any $1 \leqslant i < j \leqslant 2k+1$ the vertex $v_{i,j}$ exists in the graph. Thus, in every proper k-coloring, $v_{1,k+1}$, $v_{1,k+2}$ and $v_{2,k+2}$ would have the same color. But this is a contradiction, since $v_{1,k+1}$ and $v_{2,k+2}$ are adjacent in G. □

Theorem 4.5. *Let T be an $(n-1) \times (n-1)$ transportation matrix with no zero entry on or above the diagonal. Then, $C^* = \lceil n/2 \rceil$ and there is a simple linear time algorithm to obtain a zero-shifts plan with C^* columns.*

Proof. If n is even then $\lfloor n/2 \rfloor = \lceil n/2 \rceil$ and therefore by Lemma 4.2 $C^* = n/2$. If n is odd then from Lemmas 4.3 and 4.4 $\lfloor n/2 \rfloor < C^* \leqslant \lceil n/2 \rceil$ implying $C^* = \lceil n/2 \rceil$. The algorithm in consideration is the algorithm mentioned after Lemma 4.2. □

The following corollary is an immediate consequence of Lemma 4.1, Unger's approximation algorithm for 2ω-coloring of overlap graphs [18] and the above theorem.

Corollary 4.6. *Let T be an $(n-1) \times (n-1)$ transportation matrix, and let ω be the cardinality of a maximum clique in its corresponding overlap graph. Then, $\omega \leqslant C^* \leqslant \min\{2\omega, \lceil n/2 \rceil\}$, and there is a polynomial time algorithm to obtain a zero-shifts plan with $\min\{2\omega, \lceil n/2 \rceil\}$-columns.*

5. Coloring of overlap (circle) graphs

We turn now to discuss coloring of overlap graphs and to present a slightly better upper bound to the one obtained in [18]. Recall that given a transportation matrix T and its corresponding overlap graph G, C^* and χ (the coloring number of G), are the same. Hence, the upper bound stated in Corollary 4.6 serves as an upper bound for χ. Now, if G is an overlap graph, a natural question that comes to mind is: what is the meaning of n? We answer this question below. Gavril [10] proved that overlap graphs are equivalent to circle graphs, where a graph is a *circle graph* if the vertices can be mapped to chords of a circle so that two vertices are adjacent if and only if the corresponding chords of the circle intersect. Circle graphs were recently characterized by Bouchet [7] in terms of obstructions. As was shown by Bouchet [6] and independently by Gabor et al. [13] a circle graph is not uniquely represented on the circle. Given a circular ordering that correctly represents G, one can construct the appropriate intervals [10]. Now, given a set of intervals that correctly represents the vertices of G, we let p be the total number of end-points of these intervals. Note that there is a $(p-1) \times (p-1)$ transportation matrix T which corresponds to the set of the intervals. Hence, p can replace n in the upper bound. Observe that p might get different values for different representations of G as a set of intervals. Recall that calculating ω (the cardinality of a maximum clique in G) can be done in $O(|V|^2)$ time [15]. Given an overlap graph, a circular ordering that correctly represents G can be found in $O(|V|^2)$ time by using Spinard's algorithm [17]. Given this circular ordering, the

appropriate intervals can be constructed in linear time [10]. Therefore, $\min\{2\omega, \lceil p/2 \rceil\}$ can be calculated in $O(|V|^2)$ time. It will be nice to find a representation of G in which p is as small as possible.

Consider $\mathscr{I}$, a set of intervals on a line with 1 (p) as the first (last) end-point, and such that for any $1 \leqslant i < j \leqslant p$ there is an $[i,j]$ interval in the set. Then G_p, the overlap graph of $\mathscr{I}$, is called the *complete overlap graph* on p end-points. Note that for any 1_n transportation matrix corresponds a complete overlap graph. Also, one can verify that $\omega = \lfloor p/2 \rfloor$ for G_p. Clearly, this implies that $\omega \leqslant \lfloor p/2 \rfloor$ for any overlap graph on p end-points. Note as well that for any G_p, $2\omega > \lceil p/2 \rceil$. However, there are cases were $2\omega < \lceil p/2 \rceil$. For example, the overlap graph of the following set of intervals: $\{[i,j]: 1 \leqslant i \leqslant p - 2, \ j = i + 2\}$, for any $p \geqslant 9$.

Now, if $\omega < \lceil p/4 \rceil$, then one can use Unger's approximation algorithm [18] to obtain a 2ω-coloring. Otherwise, one can use a simple modification of the $O(|V|) \lceil p/2 \rceil$-coloring procedure for complete overlap graphs mentioned before Lemma 4.3. Hence,

Theorem 5.1. *Given an overlap graph G, a $\min\{2\omega, \lceil p/2 \rceil\}$-coloring can be found in $O(|V|^2)$ time.*

Recall that, in a way, the 2ω-coloring obtained by Unger [18] is the best approximation, since he proved that the problem of finding a $(2\omega - 1)$-coloring is NP-complete. However, as was shown in the previous section 2ω and $\lceil p/2 \rceil$ are not comparable.

Acknowledgements

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