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Abstract

Container vessel stowage is a particularly hard combinatorial problem within the shipping industry. The currently most successful approaches decompose the problem hierarchically and first generate a master plan that handle high-level constraints and objectives such as balance and stress moments, maximization of crane utility, and minimization of crane lifts. Extra crane lifts are mainly caused by containers that overstow or block each other. In particular, it is essential that no containers in a master plan stored under and over a hatch-lid cover overstow each other. In this report we show that it is an *NP*-complete problem to generate master plans that minimize the number of these lid overstows. Since any efficient approach to container vessel stowage most likely must include a master plan, the implication of this result is that future research must focus and developing good heuristics for generating master plans or decompose the generation of master plans into further levels of abstraction.

1 Introduction

Fast, reliable, and inexpensive containerized shipping has enabled the distributed lean supply chains that drive the global economy [10]. To sustain the current economic growth, it is important to achieve better understanding of the combinatorial structure of container shipping operations. A particularly challenging problem is to generate *stowage plans* for liner vessels. A stowage plan is generated at each port of call and assigns containers to slots on the vessel. To anticipate future demands, containers to load in downstream ports are often taken into account. "Good" stowage plans are hard to generate since vessels may carry more than 12.000 containers that cannot be stacked freely due to differences in height, length, weight, dangerous-goods restrictions, and power requirements in case of refrigerated containers. The main combinatorial problem, though, is to arrange containers such that the number of crane lifts is minimized. A stowage plan may induce extra crane lifts if containers *overstow* each other. A container a overstows another container b in a stack if a is placed above b but a is destined for a later port than b. In this case, a must be restowed or *shifted* in order to unload b.

Minimizing shifts for a set of containers where each container is loaded and discharged in a specific port has been shown to be *NP*-complete when the containers are to be placed in slots of more than three uncapacitated stacks [3]. The result is more than theoretical. To our knowledge, all of the proposed "flat" optimization approaches which introduce a decision variable for each possible slot assignment or similar have turned out to be intractable in practice (e.g., [5, 8, 1]). Scalable approaches are either heuristic (e.g., [4, 6, 1]) or decompose the problem hierarchically (e.g., [11, 9, 2]). The latter category is particularly well-suited for modeling the vessel stowage problem since it has a natural two-level decomposition used by industry stowage coordinators. At the first level, coordinators generate a so-called *master plan* where containers are clustered according to load and discharge port and placed in bays such that overstowage is minimized, crane utility is maximized, and high-level requirements such as balance and stress moments are satisfied. At the second level, coordinators assign containers to specific slots on the vessel to fulfill low-level stacking rules due to for instance power requirements, dangerous-goods classes, length, height, and lashing. In practice, the hardest problem is to generate the master plan. Given a good master plan, it is often an under constrained and trivial problem to assign containers to specific slots.

The question is what the combinatorial complexity of vessel stowage is given this two-level decomposition of the problem. If we ignore the second level and focus on reducing overstowage in the master plan, a common model of the problem is to divide the vessel into a number of locations with fixed capacity and assign containers to these locations without placing them in specific slots (e.g., [11, 9, 2]). It is reasonable to ignore overstowage within the location, since we assume it to be solved by the second level. Overstowage between containers placed in two different locations separated by a hatch-lid, on the other hand, cannot be ignored because as illustrated in Figure 1, *all* containers in the location above the hatch must be removed in order to unload or load a single container from the location below the hatch. We name this kind of overstows *lid overstows*.

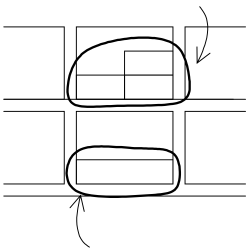

Figure 1: The ship is arriving at port p. The containers over deck lid overstows the containers under deck since the former will have to be moved when unloading (or loading) the latter.

It is easy to reduce capacitated shift minimization to lid overstow minimization. Consider a vessel where each bay has a single under and over deck location separated by a hatch-lid. If we assume that each location can hold a single container, lid overstow minimization is equivalent to capacitated shift minimization where the number of stacks equals the number of bays and the capacity of each stack is equal to 2. Unfortunately, the complexity result for shift minimization for uncapacitated stacks does not easily generalize to the capacitated case. Even for a stack capacity of 2, the problem is non-trivial and it is open whether it is *NP*-hard [3].

In this report, we prove that a slightly more elaborate version of the master plan problem is *NP*-complete. As above, we consider a model where each bay is divided into an under and over deck location separated by a hatch-lid. Each location can hold a fixed number of containers and the task is to decide how many containers to load and discharge from a location in each port. The decision must be made such that location capacities and transportation demands are satisfied and the number of lid overstows is minimized. In addition, we assume that some of the containers are *pre-placed* and cannot be moved from a given location. This requirement models that vessels seldom are empty when stowage plans are made and that load-lists may include containers with fixed slot assignments. We prove that this problem is *NP*-complete by reduction from minimum set cover.

The implication of this result is that we cannot expect tractable algorithms for generating optimal master plans given the natural model of the problem stated above. It is therefore necessary to either develop efficient heuristics and approximation algorithms for solving the master planning problem or consider further decomposition of container vessel stowage problem.

The remainder of this report is organized as follows. In Section 2, we formally define the master plan problem as the Minimum Lid Overstow Problem (MLO). We then in Section 3 remind the reader about the Minimum Set Cover Problem (MSC) and introduce a special version of this problem called MSC*. Section 4 defines a corresponding MLO instance for each MSC* instance and proves MLO to be *NP*-complete by a reduction from MSC*. Finally Section 5 draws conclucions and discuss directions for future work.

2 The Minimum Lid Overstow Problem

In this section, we introduce the Minimum Lid Overstow Problem (MLO). Described informally, the MLO is concerned with a container vessel that sails between a number of ports, picking up cargo on the way. These containers are then placed in various stowage areas on board the ship called locations. For some of the containers it is predetermined in which locations they should be placed, while we have a choice for other containers. When cargo is loaded or unloaded, i.e. when a container is moved by a crane designed for that purpose, it has a cost. The task is to assign containers to locations such that demand and capacity constraints are satisfied and the number of lid overstows are minimal. The MLO assumes that there is a single under and over deck location for each bay of the vessel. If there is cargo under deck that has to be discharged or loaded at a port, while some containers in the corresponding location over board has to stay on board, the containers over deck lid overstows the containers under deck. The MLO requires that containers can be placed such that the number of lid overstows is within a given threshold.

Key parameters of the MLO are illustrated in Figure 2. Formally, an instance of the MLO is defined by:

Instance: $\langle\, P,\, L,\, LD,\, \alpha^{\mathsf{pre}},\, \mathbf{M},\, k' \,\rangle$.

- $P \in \mathbb{N}$ is a number of ports. For convenience we let $\mathcal{P}$ denote the set of ports, i.e. $\mathcal{P} = \{1, \ldots, P\}$.
- L is the number of location columns, i.e. over and under deck location pairs. We let

$$\mathcal{L} = \mathcal{L}^{\mathsf{o}} \uplus \mathcal{L}^{\mathsf{u}} = \{l_1^{\mathsf{o}}, l_2^{\mathsf{o}}, \ldots, l_L^{\mathsf{o}}\} \uplus \{l_1^{\mathsf{u}}, l_2^{\mathsf{u}}, \ldots, l_L^{\mathsf{u}}\}$$

  denote the set of locations, where $\mathcal{L}^{\mathsf{o}}$ is the set of locations over deck and $\mathcal{L}^{\mathsf{u}}$ is the set of locations under deck.
- LD is a load-discharge $P \times P$ matrix of free containers, where $ld_{se} \in \mathbb{N}_0$ is the number of containers to transport from start port s to end port e that we have a choice of where to place. It is required that

$$\forall s \geq e \,.\, ld_{se} = 0. \tag{1}$$

  From LD we can construct the set of free transports,

$$\mathcal{T}^{\mathsf{free}} = \{(s, e) \in \mathcal{P} \times \mathcal{P} \mid ld_{se} \neq 0\}.$$

- $\alpha^{\mathsf{pre}} : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$ is an assignment of preplaced containers. It tells how many containers (possibly 0) from start port $s \in \mathcal{P}$ to end port $e \in \mathcal{P}$ that are predestined to be placed in each of the $2L$ locations. We therefore have no choice of where to put those containers.

  For an $l \in \mathcal{L}$ we identify

$$\alpha^{\mathsf{pre}}(s, e)_l = \begin{cases} \alpha^{\mathsf{pre}}(s, e)_i & \text{if } l = l_i^{\mathsf{o}} \\ \alpha^{\mathsf{pre}}(s, e)_{L+i} & \text{if } l = l_i^{\mathsf{u}} \end{cases},$$

  and $\alpha^{\mathsf{pre}}(t) = \alpha^{\mathsf{pre}}(s, e)$ if $t = (s, e)$. It is obvious that α^{pre} is given if $\alpha^{\mathsf{pre}}(s, e)_l$ is given for all $s, e \in \mathcal{P}$ and $l \in \mathcal{L}$.

  As before it is required that

$$\forall s \geq e \,.\, \alpha^{\mathsf{pre}}(s, e) = \mathbf{0}. \tag{2}$$

  From α^{pre} we can construct the set of preplaced transports similar to the construction of $\mathcal{T}^{\mathsf{free}}$:

$$\mathcal{T}^{\mathsf{pre}} = \{(s, e) \in \mathcal{P} \times \mathcal{P} \mid \alpha^{\mathsf{pre}}(s, e) \neq \mathbf{0}\}.$$

  We say that a part of a preplaced transport $t = (s, e) \in \mathcal{T}^{\mathsf{pre}}$ is placed in a location $l \in \mathcal{L}$ if $\alpha^{\mathsf{pre}}(s, e)_l \neq 0$.

  Finally we let

$$\mathcal{T} = \mathcal{T}^{\mathsf{pre}} \cup \mathcal{T}^{\mathsf{free}}$$

  be the set of all transports. We note that a transport $t \in \mathcal{T}$ can be both in $\mathcal{T}^{\mathsf{pre}}$ and $\mathcal{T}^{\mathsf{free}}$.
- $\mathbf{M} = (M^l)_{l \in \mathcal{L}}$. For $l \in \mathcal{L}$, $M^l \in \mathbb{N}$ is the maximal number of containers that can be stored at the specified location l at the same time, i.e. M^l is the maximal capacity of location l.

Concerning the capacity, the following two conditions are required to be fulfilled:

$$\forall p \in \mathcal{P}. \sum_{(s,e)\in \mathcal{T}_p^{\mathsf{on}}} \Big(ld_{se} + \sum_{l \in \mathcal{L}} \alpha^{\mathsf{pre}}(s,e)_l \Big) \ \leq \ \sum_{l\in\mathcal{L}} M^l, \tag{3}$$

where $\mathcal{T}_p^{\mathsf{on}} = \{(s,e) \in \mathcal{T} \mid s \leq p < e\}$, i.e. $\mathcal{T}_p^{\mathsf{on}}$ is the set of transports that are on board at departure from port p. (3) therefore says that there must be room enough for all containers that will be on board the ship at a specific port.

Secondly

$$\forall p \in \mathcal{P}, l \in \mathcal{L}\,.\sum_{(s,e)\in \mathcal{T}_p^{\mathsf{on}}} \alpha^{\mathsf{pre}}(s,e)_l \leq M^l, \tag{4}$$

i.e. for each location there must be room enough for the preplaced transports that will be placed in that location at a specific port.

- $k' \in \mathbb{N}_0$ is a number of allowed lid overstows, see Definition 4 below.

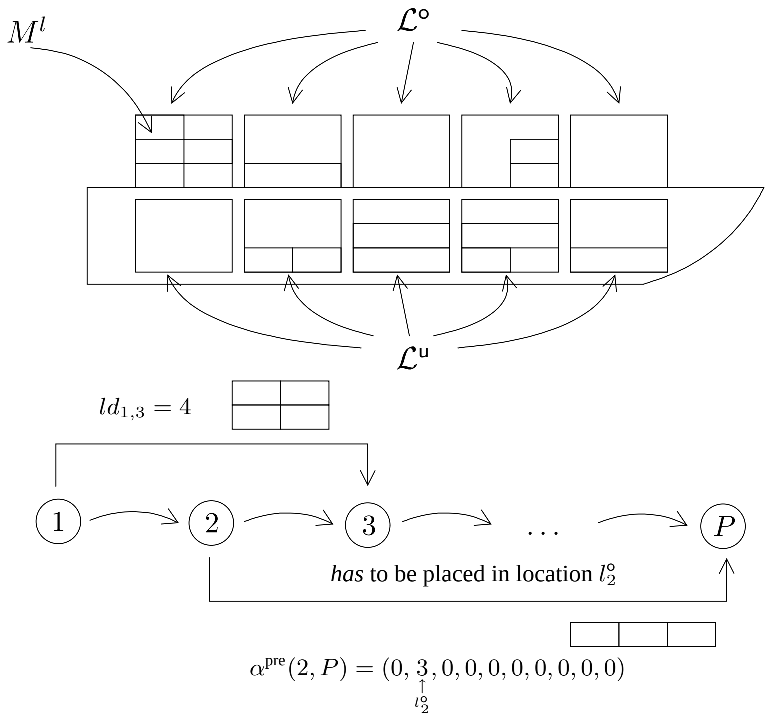

Figure 2: An illustration of the parameters of the MLO problem.

Definition 1 (Assignment) Let B be an MLO instance. An assignment for B is then a function $\alpha^{\mathsf{free}} : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$. We let $\alpha^{\mathsf{free}}(s,e)_l$ denote

$$\alpha^{\mathsf{free}}(s,e)_l = \begin{cases} \alpha^{\mathsf{free}}(s,e)_i & \text{if } l = l_i^{\mathsf{o}} \\ \alpha^{\mathsf{free}}(s,e)_{L+i} & \text{if } l = l_i^{\mathsf{u}} \end{cases} ,$$

and let $\alpha^{\mathsf{free}}(t) = \alpha^{\mathsf{free}}(s,e)$ if $t = (s,e)$. As for α^{pre}, α^{free} is said to place a part of $t \in \mathcal{T}^{\mathsf{free}}$ in l if $\alpha^{\mathsf{free}}(t)_l \neq 0$.

Clearly not all assignment corresponds to a packing of the containers in the load list. The following gives the characteristics of such an assignment:

Definition 2 (Legal assignment) Let α^{free} be an assignment for an MLO instance B. α^{free} is then a legal assignment for B if the following is true:

$$\forall (s,e) \notin \mathcal{T}^{\mathsf{free}} \,.\, \alpha^{\mathsf{free}}(s,e) = \mathbf{0}, \tag{5}$$

$$\forall (s,e) \in \mathcal{T}^{\mathsf{free}} \,.\, \sum_{l \in \mathcal{L}} \alpha^{\mathsf{free}}(s,e)_l = ld_{se}, \tag{6}$$

and

$$\forall p \in \mathcal{P}, l \in \mathcal{L} \,.\, \sum_{t \in \mathcal{T}_p^{\mathsf{on}}} \left(\alpha^{\mathsf{pre}}(t)_l + \alpha^{\mathsf{free}}(t)_l\right) \leq M^l. \tag{7}$$

(5) ensures that only transports containing containers are placed, whereas (6) ensures that all containers are placed and (7) ensures that the containers are placed within the maximal capacity of the ship.

Keeping (5) in mind when defining an assignment for an MLO instance it suffices to explicitly give $\alpha^{\mathsf{free}}(t)$ for all $t \in \mathcal{T}^{\mathsf{free}}$ and implicitly assume that $\alpha^{\mathsf{free}}(s,e) = \mathbf{0}$ for all $(s,e) \notin \mathcal{T}^{\mathsf{free}}$. We will use this later on.

When a legal assignment is available, we can consider the total assignment of containers $\alpha : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$, where $\alpha = \alpha^{\mathsf{free}} + \alpha^{\mathsf{pre}}$.

Example 3 Consider the MLO instance $B = \langle\, P,\, L,\, LD,\, \alpha^{\mathsf{pre}},\, \mathbf{M},\, k' \,\rangle$, where

- $P = 5$.
- $L = 2$, i.e. $\mathcal{L} = \{l_1^{\mathsf{o}}, l_2^{\mathsf{o}}\} \uplus \{l_1^{\mathsf{u}}, l_2^{\mathsf{u}}\}$.
- $LD = \begin{bmatrix} 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From LD we see that
$$\mathcal{T}^{\mathsf{free}} = \{(1,2), (1,4), (1,5), (2,4)\}.$$
- $\alpha^{\mathsf{pre}} : \mathcal{P} \times \mathcal{P} \to \mathbf{N}_0^{2L}$ is given by:
$$\begin{aligned} \alpha^{\mathsf{pre}}(2,3) &= (0,0,1,0) \\ \alpha^{\mathsf{pre}}(1,2) &= (0,0,0,1) \\ \alpha^{\mathsf{pre}}(3,5) &= (0,3,0,0) \end{aligned}$$
and
$$\forall\, (s,e) \notin \{(1,2), (2,3), (3,5)\} \,.\, \alpha^{\mathsf{pre}}(s,e) = \mathbf{0}.$$
The preplaced transports are the set
$$\mathcal{T}^{\mathsf{pre}} = \{(1,2), (2,3), (3,5)\}.$$
Note that $\mathcal{T}^{\mathsf{pre}} \cap \mathcal{T}^{\mathsf{free}} = \{(1,2)\}$ is non-empty.
- $M^l = 10$ for all $l \in \mathcal{L}$.
- $k' = 10$.

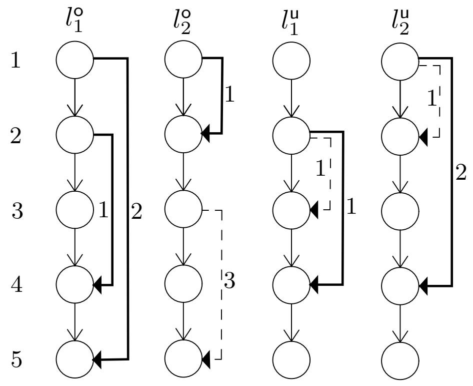

Figure 3: Each circle represents a location at a port. The arrows represents the transports in $\mathcal{T}$ and in which locations they are placed; the fat arrows are (parts of) the free transports that are placed by α^{free}, while the dotted arrows are (parts of) preplaced transports that are placed by α^{pre}. The number next to the arrow explains how many of the containers in the transport that are placed in the given location. $\mathbf{M}$ and k' are not shown in the figure.

We consider the function $\alpha^{\mathsf{free}} : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$, where the function values on $\mathcal{T}^{\mathsf{free}}$ is given by

$$\begin{aligned}
\alpha^{\mathsf{free}}(1,2) &= (0,1,0,0)\\
\alpha^{\mathsf{free}}(1,4) &= (0,0,0,2)\\
\alpha^{\mathsf{free}}(1,5) &= (2,0,0,0)\\
\alpha^{\mathsf{free}}(2,4) &= (1,0,1,0).
\end{aligned}$$

α^{free} is then a legal assignment for B and is visualized in Figure 3. $\diamond$

We will now look at the concept of lid overstows.

If some containers are placed in the over deck location l_j° at a time when other containers are getting on or off the corresponding under deck location l_j^{u}, then each container on top has to be moved back and forth, and so each container on top contributes with one lid overstow. The formal definition is given below.

Definition 4 (Lid overstow) Let α^{free} be a legal assignment for the MLO instance $B = \langle\, P,\ L,\ LD,\ \alpha^{\mathsf{pre}},\ \mathbf{M},\ k'\,\rangle$.

First we define the set of on/off ports $\mathbf{O} = (\mathcal{O}_j)_{j\in\{1,\ldots,L\}}$. This is a set of ports, where a transport is getting on or off at the under deck location l_j^{u}, that is:

$$\mathcal{O}_j = \left\{ p \in \mathcal{P} \;\middle|\; \sum_{q=1}^{P} \Big(\alpha(q,p)_{l_j^{\mathsf{u}}} + \alpha(p,q)_{l_j^{\mathsf{u}}} \Big) > 0 \right\}.$$

For each port in $\mathcal{O}_j$ there is a possibility for lid overstows while there is none at any other ports for location pair j. For a port p in $\mathcal{O}_j$ we therefore have to consider the number of containers that are placed in the over deck location l_j° at port p.

The total number of lid overstows is then

$$\#LO_{\alpha^{\mathsf{free}}} = \sum_{j=1}^{L} \sum_{(s,e)\in\mathcal{T}} \alpha(s,e)_{l_j^{\circ}} \cdot \big|\mathcal{O}_j \cap\,]\, s,e\, [\, \big|.$$

By this we can talk about the lid overstows caused by a transport $(s,e) \in \mathcal{T}$ as

$$\#LO_{\alpha^{\mathsf{free}}}(s,e) = \sum_{j=1}^{L} \alpha(s,e)_{l_j^{\circ}} \cdot \big|\, \mathcal{O}_j \cap\,]\, s,e\, [\, \big|, \tag{8}$$

and obviously

$$\#LO_{\alpha^{\text{free}}} = \sum_{(s,e)\in\mathcal{T}} \#LO_{\alpha^{\text{free}}}(s,e).$$

Example 5 Consider the MLO instance described in Example 3. As seen from the definition of lid overstows, when counting the number of lid overstows for an assignment α^{free}, we are only interested in the ports where there is a load- or discharge to or from an under deck location, i.e. $\mathcal{O}_j$.

Counting the number of lid overstows is now very simple: First we fill the circles corresponding to ports in $\mathcal{O}_j$. Then we just have to take each transport placed in an over deck location and count the number of filled circles in the corresponding under deck location between (excluding) the load port and the discharge port of the transport. This is then multiplied with the number of containers in the transport that are placed in the considered location and finally we add all those numbers and get the number of lid overstows.

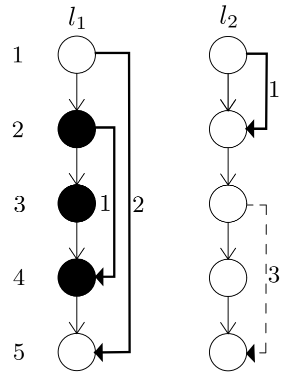

Figure 4: For each pair of locations (l_j^{o} and l_j^{u}) the figure shows a column. The parts of transports that are placed in the over deck location l_j^{o} is shown in column j while the ports of $\mathcal{O}_j$ are filled in column j (compare with Figure 3).

For this particular example (for reference see Figure 4) we have that the number of lid overstows is

$$\begin{aligned}
\#LO_{\alpha^{\text{free}}} &= \sum_{(s,e)\in\mathcal{T}} \sum_{j=1}^{L} \alpha(s,e)_{l_j^{\text{o}}} \cdot \left|\, \mathcal{O}_j \cap \,]\, s,e\, [\, \right| \\
&= \alpha(1,5)_{l_1^{\text{o}}} \cdot \left|\{2,3,4\}\right| + \alpha(2,4)_{l_1^{\text{o}}} \cdot \left|\{3\}\right| + \alpha(1,2)_{l_2^{\text{o}}} \cdot \emptyset + \alpha(3,5)_{l_2^{\text{o}}} \cdot \left|\{4\}\right| \\
&= 2\cdot 3 + 1\cdot 1 + 1\cdot 0 + 3\cdot 1 = 10.
\end{aligned}$$

$\diamond$

We can now define the Minimum Lid Overstow Problem:

Definition 6 (Minimum Lid Overstow Problem) The Minimum Lid Overstow Problem is given by the following:

Instance: $\langle\, P,\, L,\, LD,\, \alpha^{\text{pre}},\, \mathbf{M},\, k'\,\rangle$.
Question: Does there exist an assignment α^{free} that causes k' or less lid overstows, i.e. where $\#LO_{\alpha^{\text{free}}} \leq k'$?

3 Minimum Set Cover

In the following the *NP*-completeness of the MLO-problem will be proven. For this we need another known *NP*-complete problem that can be reduced to MLO. A version of the Minimum Set Cover Problem (MSC) will be used for this purpose.

Definition 7 (Minimum Set Cover Problem) The Minimum Set Cover Problem is given as follows:

Instance: $\langle\, \mathcal{S},\, \mathcal{C},\, k\,\rangle$.

- $\mathcal{S}$ is a finite set,
- $\mathcal{C} \subseteq 2^{\mathcal{S}}$ is a set of subsets of $\mathcal{S}$, i.e. $\mathcal{C}' \in \mathcal{C} \Rightarrow \mathcal{C}' \subseteq \mathcal{S}$, and
- $k \in \mathbb{N}$ is a number with $k \leq |\mathcal{C}|$.

Question: Does $\mathcal{C}$ contain a cover for $\mathcal{S}$ of size less or equal to k?

I.e. we consider the existence of a set $\mathcal{K} \subseteq \mathcal{C}$, such that

$$|\mathcal{K}| \leq k \quad \text{and} \quad \forall s \in \mathcal{S} \,.\, \exists \mathcal{C}' \in \mathcal{K} \,.\, s \in \mathcal{C}'.$$

Example 8 Let $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$, and let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$, where $\mathcal{C}_1 = \{s_1, s_2, s_3\}$, $\mathcal{C}_2 = \{s_2, s_5\}$ and $\mathcal{C}_3 = \{s_3, s_4, s_5\}$ (see Figure 5) and let $k = 2$.

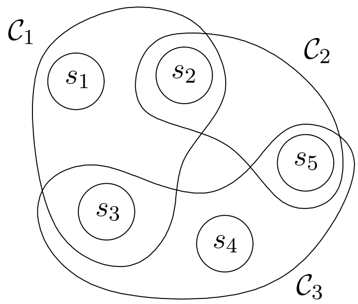

Figure 5: $\mathcal{S}$ and $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$

There is no cover of size 1, since neither $\mathcal{C}_1$, $\mathcal{C}_2$ nor $\mathcal{C}_3$ contains all elements. $\mathcal{C}_1$ and $\mathcal{C}_3$ is a cover for $\mathcal{S}$ of size 2 and is the only cover of size 2.

We can conclude that $\mathcal{S}$ is a "'yes"'-instance. $\diamond$

This decision problem is known to be *NP*-complete [7].

We will use a variation of this problem, MSC* to show the *NP*-completeness of MLO. The only difference between MSC and MSC* is that we for MSC* require that each instance of the problem *does* have a cover, i.e. that $\mathcal{C}$ in it self is a cover of $\mathcal{S}$. Since it can be determined in polynomial time whether there exists a cover of any size or not, we can make a polynomial reduction of MSC to MSC*:

An instances of MSC *with* a cover is an MSC* instance and is simply mapped into itself, while an instance of MSC without a cover is mapped into a generic "'no"'-instance of MSC*, e.g. $\langle\, \mathcal{S} = \{s_1, s_2\},\, \mathcal{C} = \{\{s_1\}, \{s_2\}\},\, k = 1\,\rangle$. From this it follows that MSC* is *NP*-complete as well.

4 Reducing MSC* to MLO

The key idea of the reduction from MSC* to MLO is to associate each element of $\mathcal{S}$ with a *local element transport* of a single container and associate each subset in $\mathcal{C}$ with an over deck location with a capacity of one container. We can now define the preplaced containers such that they occupy all the space under deck and are loaded and unloaded in such a way that a local transport of $s_i \in \mathcal{S}$ only lid overstows a preplaced container in the over deck location associated with $\mathcal{C}_j \in \mathcal{C}$ if $s_i \notin \mathcal{C}_j$. In this way, if local transports are placed such that they do not lid overstow any containers, it must be the case that the set of subsets in $\mathcal{C}$ corresponding to over deck locations holding some containers must cover $\mathcal{S}$. This cover, however, is not guaranteed to be minimal. To ensure this, we introduce a*blocker transport*. The blocker transport goes from the first to the last port. The number of blocker containers equal the number of subsets in $\mathcal{C}$, and we arrange the preplaced containers such that the blocker transport induces fewest lid overstows if placed in the over deck locations associated with the subsets in $\mathcal{C}$. For each of these, however, we introduce a new bay, where a blocker transport can be placed with a slight overhead in the number of lid overstows. This construction ensures that local transports try to use as few over deck locations as possible and thus form a minimum set cover. A small example of the reduction is shown in Figure 8.

In order to make a reduction from MSC* to MLO, we introduce a corresponding MLO instance for each MSC* instance. For this purpose, we need to order the elements of $\mathcal{S}$, such that each element s_i of $\mathcal{S}$ is uniquely determined by its index i. In order to do this we just make a set isomorphism from $\mathcal{S} \xrightarrow{\sigma} \{1, 2, \ldots, |\mathcal{S}|\}$ and use the notation s_i for the element $\sigma^{-1}(i)$. Likewise we assume that any element $\mathcal{C}_j$ of $\mathcal{C}$ is uniquely determined by its index j.

Definition 9 (Corresponding MLO) Let A be an MSC* instance and define the corresponding MLO instance by specifying P, L, LD, α^{pre}, $\mathbf{M}$ and k' as follows:

- $P = 6|\mathcal{S}| + 2$.
- $L = 2|\mathcal{C}|$.
- The non-zero entries of LD are:
$$ld_{1P} = |\mathcal{C}|$$
and
$$ld_{3i-2,3i+1} = 1 \quad \text{for all } i \in \{1, \ldots, |\mathcal{S}|\}.$$
We therefore have that
$$\mathcal{T}^{\mathsf{free}} = \big\{(3i-2, 3i+1) \;\big|\; 1 \leq i \leq |\mathcal{S}|\big\} \;\cup\; \big\{(1, P)\big\}.$$
We will distinguish between the free transports and let $b = (1, P) \in \mathcal{T}^{\mathsf{free}}$ be called the blocker transport and for all $i \in \{1, \ldots, |\mathcal{S}|\}$ let $e_i = (3i-2, 3i+1) \in \mathcal{T}^{\mathsf{free}}$ be called an element transport; we create an element transport for each element $s_i \in \mathcal{S}$ and the transport e_i is therefore associated with the element $s_i \in \mathcal{S}$.
- Preplaced containers will only be placed in under deck locations. α^{pre} is defined in terms of $\mathcal{O}_j$, where
$$\begin{aligned}\mathcal{O}_j = \quad & \Big\{\{3i-1, 3i\} \;\Big|\; i \in \{1, \ldots, |\mathcal{S}|\} \text{ and } s_i \notin \mathcal{C}_j\Big\} \\ & \cup \\ & \Big\{\{3i-1+3|\mathcal{S}|, 3i+3|\mathcal{S}|\} \;\Big|\; i \in \{1, \ldots, |\mathcal{S}|\} \text{ and } s_i \in \mathcal{C}_j\Big\} \\ & \cup \\ & \{1, P\}\end{aligned}$$
for $j \leq |\mathcal{C}|$, and
$$\begin{aligned}\mathcal{O}_j = \quad & \Big\{\{3i-1, 3i\} \;\Big|\; i \in \{1, \ldots, |\mathcal{S}|\}\Big\} \\ & \cup \\ & \{1, P-1, P\}\end{aligned}$$
for $j > |\mathcal{C}|$.
A visualization of $\mathcal{O}_j$ is made in Figure 6.
We then have:
$$\alpha^{\mathsf{pre}}(s, e)_l = \begin{cases} 1 & \text{if } l = l^{\mathsf{u}}_j, s \in \mathcal{O}_j \setminus \{P\} \text{ and } e = \min\{o \in \mathcal{O}_j \mid s < o\} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$
- We define the maximal capacity simply as
$$\forall l \in \mathcal{L} \,.\, M^l = 1.$$
- $k' = 2|\mathcal{C}||\mathcal{S}| + k$.

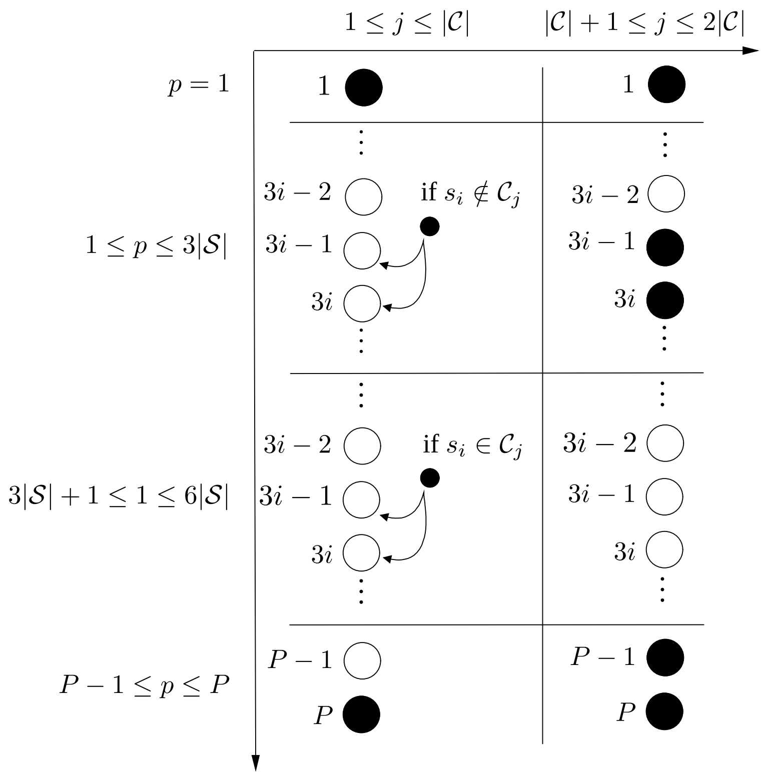

Figure 6: The figure shows how the elements of $\mathcal{O}_j$ are distributed according to the index j and the port p. A filled (black) circle means that the corresponding port p is in $\mathcal{O}_j$ for the corresponding j. As the figure suggests, some of the circles are filled depending on the elements of $\mathcal{C}_j$ while others are filled under any circumstances.

It is an easy exercise to show that (1) and (2) are fulfilled and that $\mathcal{T} = \mathcal{T}^{\mathsf{free}} \uplus \mathcal{T}^{\mathsf{pre}}$, i.e. the set of free transports and the set of preplaced transports are disjoint (see Lemma 10 and Lemma 11). It should likewise be easy to see that the "'capacity constraints"', i.e. (3) and (4), are met, since there are at most $|\mathcal{C}|+1$ containers from free transports on board at any time, while there per construction is at most one preplaced container per under deck location at a time, see Lemma 10.

Lemma 10 The corresponding MLO is an MLO.

Proof: (1) is obvious since if $ld_{se} \neq 0$, then either $(s,e) = (1,P)$ or $(s,e) = (3i-2, 3i+1)$, and in either case $s < e$.

(2) is just as obvious, since:

$$\begin{aligned} \alpha^{\mathsf{pre}}(s,e) \neq \mathbf{0} \quad &\Rightarrow \quad \exists j \,.\, \alpha^{\mathsf{pre}}(s,e)_{l_j^{\mathsf{u}}} = 1 \\ &\Rightarrow \quad \exists j \,.\, e \in \{o \in \mathcal{O}_j \mid s < o\} \quad \Rightarrow \quad s < e. \end{aligned}$$

(4) is true per construction since the preplaced transports are defined as transports between 2 consecutive ports in the set $\mathcal{O}_j$; if p and l are given and the transport $(s,e) \in \mathcal{T}^{\mathsf{pre}} \cap \mathcal{T}_p^{\mathsf{on}}$ is placed (partly) in l, i.e. $s \leq p < e$ and $\alpha^{\mathsf{pre}}(s,e)_l \neq 0$, then no other preplaced transport $(m',n') \in \mathcal{T}^{\mathsf{pre}} \cap \mathcal{T}_p^{\mathsf{on}}$ is placed in location l. Each preplaced transport only contains one container, so

$$\sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}} \alpha^{\mathsf{pre}}(s,e)_l \stackrel{*}{=} \sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{pre}}} \alpha^{\mathsf{pre}}(s,e)_l \quad \leq \quad \alpha^{\mathsf{pre}}(s',e')_l \quad = \quad 1$$

for an $(s', e') \in \mathcal{T}^{\mathsf{pre}}$. $*$ is true since $\alpha^{\mathsf{pre}}(s,e) = \mathbf{0}$ for all $(s,e) \notin \mathcal{T}^{\mathsf{pre}}$.

(3) also follows easily. Let $p \in \mathcal{P}$ be given. If $(s,e) \in \mathcal{T}_p^{\mathsf{on}} \cap \mathcal{T}^{\mathsf{free}}$, then $s \leq p < e$ and we must have that $(s,e) \in \{e_{\lceil \frac{p}{3} \rceil}, b\}$. Hereby we get

$$\sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{free}}} ld_{se} \leq ld_{3\lceil \frac{p}{3}\rceil-2,3\lceil \frac{p}{3}\rceil+1} + ld_{1P} = 1 + |\mathcal{C}|.$$

If $(s,e) \in \mathcal{T}_p^{\mathsf{on}} \cap \mathcal{T}^{\mathsf{pre}}$ we have as argued above, that

$$\begin{aligned}\sum_{l\in\mathcal{L}} \sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{pre}}} \alpha^{\mathsf{pre}}(s,e)_l &\stackrel{*}{=} \sum_{l\in\mathcal{L}^{\mathsf{u}}} \sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{pre}}} \alpha^{\mathsf{pre}}(s,e)_l \\ &\leq \sum_{l\in\mathcal{L}^{\mathsf{u}}} 1 = L = 2|\mathcal{C}|,\end{aligned}$$

where $*$ is true since $\alpha^{\mathsf{pre}}(t)_{l_j^{\circ}} = 0$ for all $t \in \mathcal{T}$ and all j. Since $\mathcal{T}^{\mathsf{pre}}$ and $\mathcal{T}^{\mathsf{free}}$ are disjoint we have that if $(s,e) \in \mathcal{T}^{\mathsf{free}}$, then $(s,e) \notin \mathcal{T}^{\mathsf{pre}}$ and hereby $\alpha^{\mathsf{pre}}(s,e) = \mathbf{0}$. Likewise if $(s,e) \in \mathcal{T}^{\mathsf{pre}}$, then $(s,e) \notin \mathcal{T}^{\mathsf{free}}$ and $\alpha^{\mathsf{free}}(s,e) = \mathbf{0}$. This finally gives us:

$$\begin{aligned}&\sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}} \Big(ld_{se} + \sum_{l\in\mathcal{L}} \alpha^{\mathsf{pre}}(s,e)_l \Big) \\ &\quad= \sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{free}}} ld_{se} + \sum_{(s,e)\in\mathcal{T}_p^{\mathsf{on}}\cap\mathcal{T}^{\mathsf{pre}}} \sum_{l\in\mathcal{L}} \alpha^{\mathsf{pre}}(s,e)_l \\ &\quad= 1 + |\mathcal{C}| + 2|\mathcal{C}| \leq 4|\mathcal{C}|,\end{aligned}$$

i.e. (3) is true. □

Lemma 11 For a corresponding MLO the following is true:

- $\mathcal{T}^{\mathsf{free}} \cap \mathcal{T}^{\mathsf{pre}} = \emptyset$.
- The $\mathcal{O}_j$ given in the definition of the corresponding MLO equals the $\mathcal{O}_j$ given in the definition of lid overstows.

Proof: Notice that the first element of a free transport belongs to residue class 1 modulo 3, while the only elements in any $\mathcal{O}_j$ with this property are 1 and $P - 1 = 6|\mathcal{S}| + 1$. From this we gather:

$$\begin{aligned}(s,e) \in \mathcal{T}^{\mathsf{pre}} \quad &\Rightarrow \quad \exists j \,.\, s \in \mathcal{O}_j \setminus \{P\} \text{ and } e = \min\{p \in \mathcal{O}_j \mid s < p\} \\ &\Rightarrow \quad [s]_3 = [1]_3 \text{ or } [s]_3 \neq [1]_3 \\ &\Rightarrow \quad (s = 1 \text{ and } [e]_3 = [2]_3) \text{ or } (s = P-1 \text{ and } d = P) \text{ or } [s]_3 \neq 1 \\ &\Rightarrow \quad (s,e) \notin \mathcal{T}^{\mathsf{free}},\end{aligned}$$

hence $\mathcal{T}^{\mathsf{pre}} \cap \mathcal{T}^{\mathsf{free}} = \emptyset$.

Last we show that the $\mathcal{O}_j$ given in the definition of the corresponding MLO does equal the $\mathcal{O}_j$ given in the definition of lid overstows. To be able to distinguish between the two we will in the following let $\mathcal{O}'_j$ denote the set given in the definition of the corresponding MLO, while $\mathcal{O}_j$ denotes the set given in the definition of lid overstow. For a given j we then get

$$\begin{aligned}p \in \mathcal{O}_j \quad &\Rightarrow \quad \exists q \,.\, \alpha(q,p)_{l_j^{\mathsf{u}}} \neq 0 \text{ or } \exists q \,.\, \alpha(p,q)_{l_j^{\mathsf{u}}} \neq 0 \\ &\stackrel{(9)}{\Rightarrow} \quad p, q \in \mathcal{O}'_j,\end{aligned}$$

i.e. $\mathcal{O}_j \subseteq \mathcal{O}'_j$. On the other hand,

$$
\begin{aligned}
p \in \mathcal{O}'_j \Rightarrow & \left\{ \begin{array}{c} p \neq P \text{ and } \exists q \in \mathcal{O}'_j \,.\, \alpha^{\mathsf{pre}}(p,q)_{l^{\mathsf{u}}_j} = 1 \\ \text{or} \\ p = P \text{ and } \exists q \in \mathcal{O}'_j \,.\, \alpha^{\mathsf{pre}}(q,p)_{l^{\mathsf{u}}_j} = 1 \end{array} \right. \\
& \Rightarrow \sum_{q'=1}^{P} \left(\alpha(q',p)_{l^{\mathsf{u}}_j} + \alpha(p,q')_{l^{\mathsf{u}}_j} \right) > 0 \\
& \Rightarrow p \in \mathcal{O}_j,
\end{aligned}
$$

i.e. $\mathcal{O}'_j \subseteq \mathcal{O}_j$. That is, $\mathcal{O}_j = \mathcal{O}'_j$. $\square$

Example 12 Let $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$, and let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ and $k = 2$, where $\mathcal{C}_1 = \{s_1, s_2, s_3\}$, $\mathcal{C}_2 = \{s_2, s_5\}$ and $\mathcal{C}_3 = \{s_3, s_4, s_5\}$ as in Example 8.

The corresponding MLO is visualized in Figure 7.

Since all preplaced transports are placed in under deck locations, the figure only shows those locations to simplify the drawing. As in Figure 4 a circle that corresponds to a location l^{u}_j and a port p is filled (black) if $p \in \mathcal{O}_j$.

The free transports $(s,e) \in \mathcal{T}^{\mathsf{free}}$ are shown as arrows from the level of port s to the level of port e, since we yet have no assignment to place them. They are labeled with their name followed by the number of containers in the transport. Only the load- and discharge ports for the preplaced transports are shown, this is $\mathbf{O}$. The preplaced transports go from a filled circle in a column to the next filled circle below in the same column all containing one container.

To make the drawing more clear, the number of the ports are left out, but starts with 1 in the top and ends with 32 in the bottom.

$k' = 32$. The capacity is as mentioned 1 for all locations. $\diamond$

In order to show the *NP*-completeness of MLO, we will show that an MSC* instance is a "'yes"' instance if and only if the corresponding MLO instance is a "'yes"' instance. To do that we introduce the corresponding MLO assignment for an MSC* cover.

Definition 13 (Corresponding assignment) Let $A = \langle\, \mathcal{S},\, \mathcal{C},\, k \,\rangle$ be an MSC* instance, and let B be the corresponding MLO instance. Let $\mathcal{K}$ be a cover for $\mathcal{S}$.
We then define the corresponding assignment $\alpha^{\mathsf{free}}_{\mathcal{K}} : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$ for B as follows:

$$
\forall i \in \{1, \ldots, |\mathcal{S}|\} \,.\, \alpha^{\mathsf{free}}_{\mathcal{K}}(e_i)_j = \left\{ \begin{array}{ll} 1 & \text{if } j = \min\left\{ j' \leq |\mathcal{C}| \;\middle|\; s_i \in \mathcal{C}_{j'} \text{ and } \mathcal{C}_{j'} \in \mathcal{K} \right\} \\ 0 & \text{otherwise} \end{array} \right. , \tag{10}
$$

$$
\alpha^{\mathsf{free}}_{\mathcal{K}}(b)_j = \left\{ \begin{array}{ll} 1 & \text{if } j \leq |\mathcal{C}| \text{ and } \mathcal{C}_j \notin \mathcal{K} \\ 1 & \text{if } |\mathcal{C}| + 1 \leq j \leq |\mathcal{C}| + |\mathcal{K}| \\ 0 & \text{otherwise} \end{array} \right. \tag{11}
$$

and

$$
\forall (s,e) \notin \mathcal{T}^{\mathsf{free}} \,.\, \alpha^{\mathsf{free}}_{\mathcal{K}}(s,e) = \mathbf{0}. \tag{12}
$$

The corresponding assignment places as seen an element transport e_i in an over deck location l^{o}_j where the corresponding $\mathcal{C}_j$ is in $\mathcal{K}$ and contains the associated element s_i. If there are more sets in the cover that contains the element s_i, then the assignments places the element transport in the location corresponding to the first of those sets.

The blocker transport b is placed with one container in each of the first $|\mathcal{C}|$ over deck locations where the corresponding $\mathcal{C}_j$'s are *not* in the cover (i.e. that are not already occupied by an element transport). The remaining part of the blocker transport is then placed with one container in other over deck locations (as many as needed, starting with $l^{\mathsf{o}}_{|\mathcal{C}|+1}$ and continuing consecutively).

The following proposition shows that $\alpha^{\mathsf{free}}_{\mathcal{K}}$ is a legal assignment:

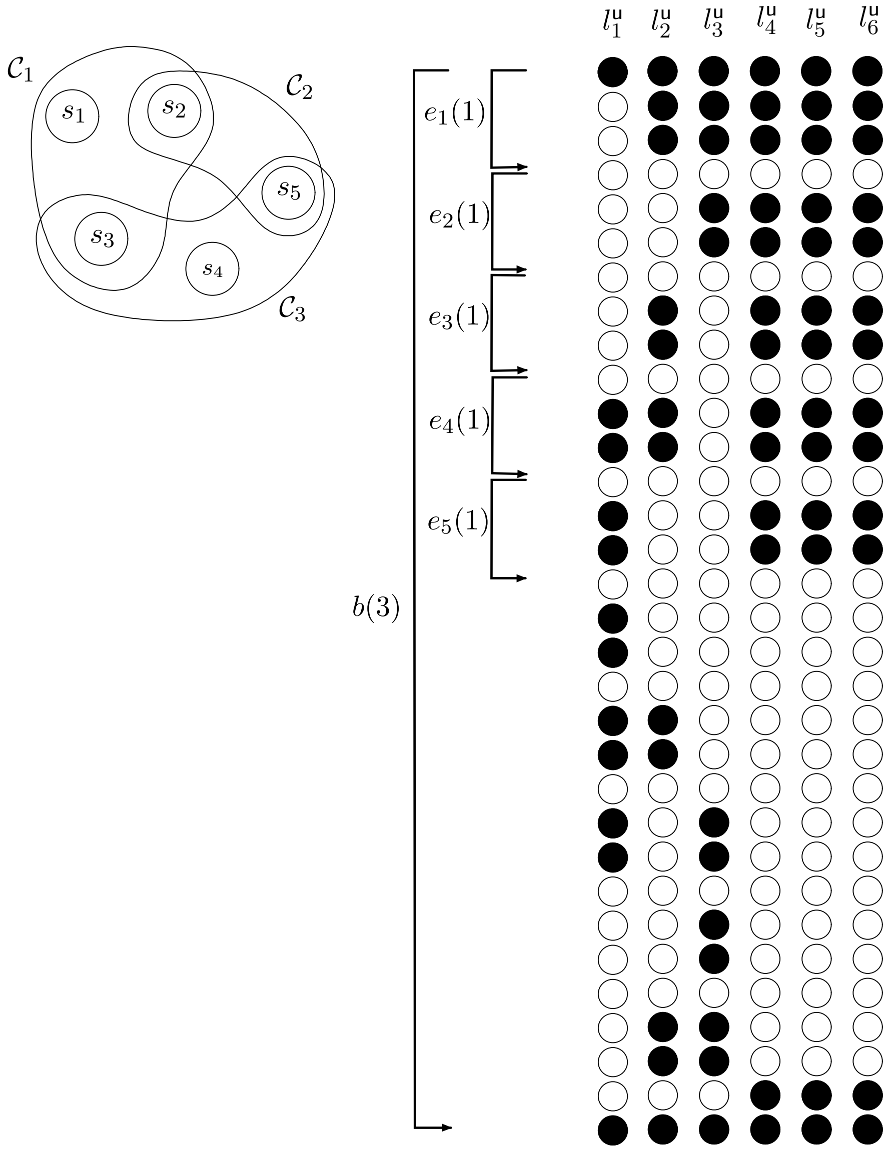

Figure 7: An MSC* instance and the corresponding MLO instance.

Proposition 14 Let A be an MSC* instance with a cover $\mathcal{K}$, and let B be the corresponding MLO instance. Then the corresponding assignment $\alpha_{\mathcal{K}}^{\mathsf{free}}$ is a legal assignment for B.

Proof: We will show that $\alpha_{\mathcal{K}}^{\mathsf{free}}$ fulfills (5), (6) and (7) since the requirement in Definition 1 obviously is fulfilled.

(5) is true by definition.

(6): First we will consider the element transports. Let therefore i be given. By inspecting the definition of $\alpha_{\mathcal{K}}^{\mathsf{free}}$ we see that the only element of $\alpha_{\mathcal{K}}^{\mathsf{free}}(e_i)$ that is non-zero is $\phi_{\mathcal{K}}(e_i)_j$, where $j = \min\{j' \leq |\mathcal{C}| \mid s_i \in \mathcal{C}_{j'} \text{ and } \mathcal{C}_{j'} \in \mathcal{K}\}$. The mentioned set is non-empty since $\mathcal{K}$ is a cover for $\mathcal{S}$, which means that there exists a set in $\mathcal{K}$, that contains s_i. Since the set is finite as well there exists exactly one minimum. This means that exactly one element of $\alpha_{\mathcal{K}}^{\mathsf{free}}$ is 1 while the others are zero, i.e.

$$\sum_{l\in\mathcal{L}} \alpha_{\mathcal{K}}^{\mathsf{free}}(e_i)_l = 1 = ld_{3i-2,3i+1},$$

as required.

If we instead consider the blocker transport b, we see that the elements of $\alpha_{\mathcal{K}}^{\mathsf{free}}(b)$ that equals 1 are $\alpha_{\mathcal{K}}^{\mathsf{free}}(b)_j$'s with

$$j \in \{j' \leq |\mathcal{C}| \mid \mathcal{C}_{j'} \notin \mathcal{K}\} \cup \{j' \mid |\mathcal{C}|+1 \leq j' \leq |\mathcal{C}|+|\mathcal{K}|\},$$

while the rest of the elements equals zero. However

$$\big|\{j' \leq |\mathcal{C}| \mid \mathcal{C}_{j'} \notin \mathcal{K}\}\big| = |\mathcal{C}| - |\mathcal{K}| \text{ and } \big|\{j' \mid |\mathcal{C}|+1 \leq j' \leq |\mathcal{C}|+|\mathcal{K}|\}\big| = |\mathcal{K}|,$$

so

$$\sum_{l\in\mathcal{L}} \alpha_{\mathcal{K}}^{\mathsf{free}}(b)_l = |\mathcal{C}| - |\mathcal{K}| + |\mathcal{K}| = |\mathcal{C}| = ld_{1P},$$

again as required.

Finally we will show (7). Let therefore p and l be given. First we will consider the case where $l \in \mathcal{L}^{\mathsf{u}}$. We have (by inspection) that $\alpha_{\mathcal{K}}^{\mathsf{free}}(t)_l = 0$ for all $t \in \mathcal{T}$, which gives:

$$\sum_{t\in\mathcal{T}_p^{\mathsf{on}}} \left(\alpha^{\mathsf{pre}}(t)_l + \alpha_{\mathcal{K}}^{\mathsf{free}}(t)_l\right) = \sum_{t\in\mathcal{T}_p^{\mathsf{on}}} \alpha^{\mathsf{pre}}(t)_l \overset{*}{\leq} M^l,$$

where $*$ is true since the corresponding MLO is an MLO and therefore fulfill the capacity criteria (4).

If on the other hand $l = l_j^{\mathsf{o}} \in \mathcal{L}^{\mathsf{o}}$ then $\alpha^{\mathsf{pre}}(t)_l = 0$ for all $t \in \mathcal{T}$. If an element transport e_i is placed in l, i.e. $\alpha_{\mathcal{K}}^{\mathsf{free}}(e_i)_l = 1$, then $\mathcal{C}_j \in \mathcal{K}$ and then $\alpha_{\mathcal{K}}^{\mathsf{free}}(b)_l \neq 1$. Likewise

$$\alpha_{\mathcal{K}}^{\mathsf{free}}(b)_l = 1 \quad \Rightarrow \quad \mathcal{C}_j \notin \mathcal{K} \quad \Rightarrow \quad \alpha_{\mathcal{K}}^{\mathsf{free}}(e_i)_l \neq 1.$$

Since two element transports are not on board the ship at the same time, we gather that

$$\sum_{t\in\mathcal{T}_p^{\mathsf{on}}} \left(\alpha^{\mathsf{pre}}(t)_l + \alpha^{\mathsf{free}}(t)_l\right) = \sum_{t\in\mathcal{T}_p^{\mathsf{on}}} \alpha^{\mathsf{free}}(t)_l \leq 1.$$

That is, (7) is true in this case too. □

Example 15 Consider the MSC* instance A defined in Example 8 and the corresponding MLO, as presented in Example 12.
Let the cover $\mathcal{K}$ be $\mathcal{K} = \{\mathcal{C}_1, \mathcal{C}_3\}$. Then the corresponding assignment $\alpha_{\mathcal{K}}^{\mathsf{free}}$ is the function

$$\alpha_{\mathcal{K}}^{\mathsf{free}} : \{1,\ldots,32\} \times \{1,\ldots,32\} \to \mathbb{N}_0^{12}$$

where the function values on $\mathcal{T}^{\mathsf{free}}$ are given by:

$$\begin{array}{rll}
\alpha_{\mathcal{K}}^{\mathsf{free}}(e_1) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(1,4) & = (1,0,0,0,0,0,0,0,0,0,0,0) \\
\alpha_{\mathcal{K}}^{\mathsf{free}}(e_2) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(4,7) & = (1,0,0,0,0,0,0,0,0,0,0,0) \\
\alpha_{\mathcal{K}}^{\mathsf{free}}(e_3) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(7,10) & = (1,0,0,0,0,0,0,0,0,0,0,0) \\
\alpha_{\mathcal{K}}^{\mathsf{free}}(e_4) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(10,13) & = (0,0,1,0,0,0,0,0,0,0,0,0) \\
\alpha_{\mathcal{K}}^{\mathsf{free}}(e_5) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(13,16) & = (0,0,1,0,0,0,0,0,0,0,0,0) \\
\alpha_{\mathcal{K}}^{\mathsf{free}}(b) & = \alpha_{\mathcal{K}}^{\mathsf{free}}(1,32) & = (0,1,0,1,1,0,0,0,0,0,0,0).
\end{array}$$

$\alpha_{\mathcal{K}}^{\mathsf{free}}$ is visualized in Figure 8. The figure only shows one location/column for each bay. This should cause no confusion since the preplaced transports are placed in under deck locations only, while the free transports are placed in over deck locations only. ◇

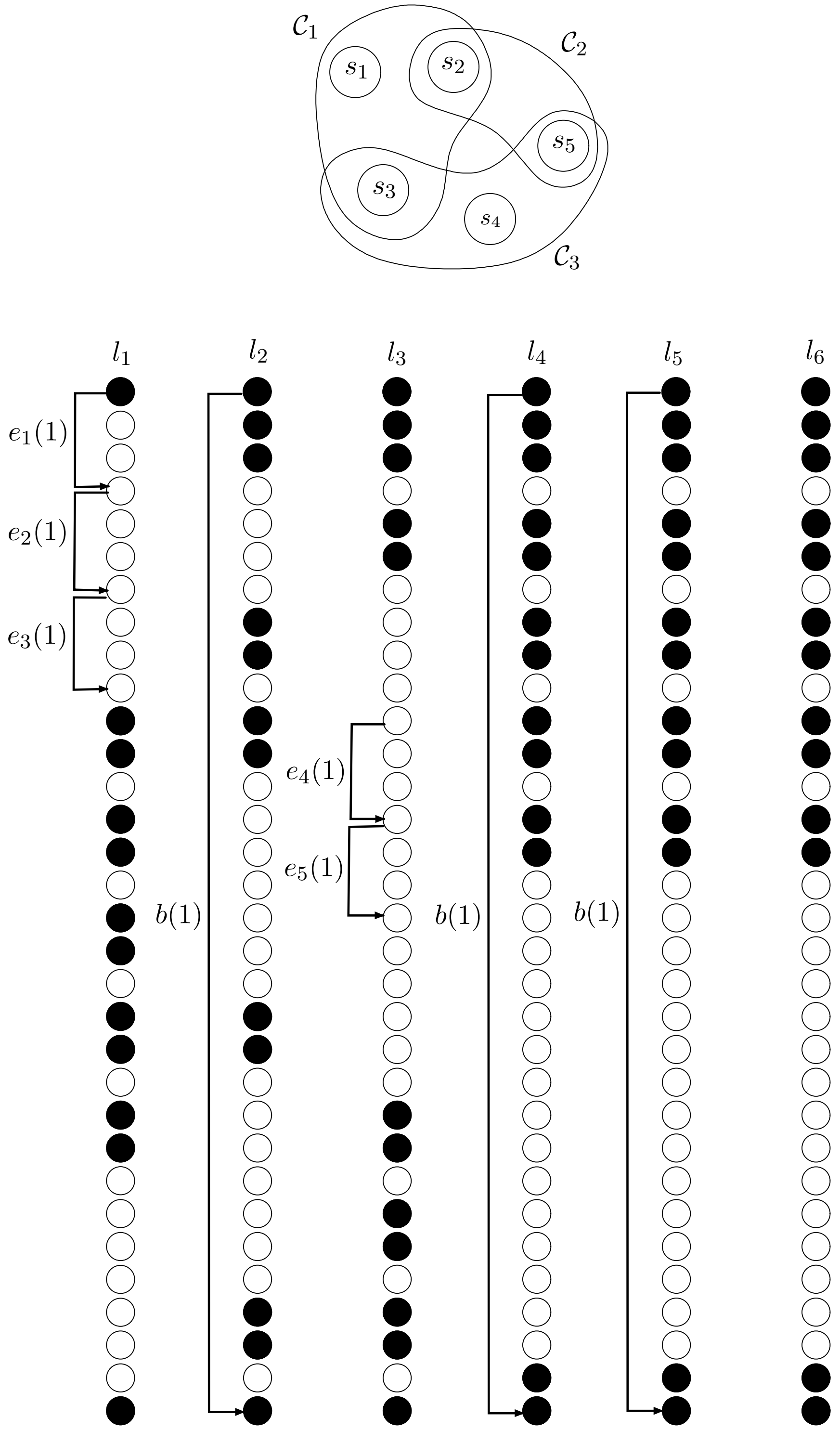

Figure 8: A corresponding MLO instance with the assignment $\alpha_{\mathcal{K}}^{\mathsf{free}}$ corresponding to the cover $\mathcal{K} = \{\mathcal{C}_1, \mathcal{C}_3\}$.

For a corresponding MLO, the following facts are easily deduced from the definition.

Proposition 16 Let A be an MSC* instance and let B be the corresponding MLO instance. Let α^{free} be an arbitrary, legal assignment for B.
The following is then true:

1. α^{free} places free transports $t \in \mathcal{T}^{\mathsf{free}}$ in over deck locations only. If part of the blocker transport is placed in an over deck location l_j^{o} then no other transport is placed in l_j^{o}.

2. Each element transport e_i is placed in exactly one location. The blocker transport b is placed in exactly $|\mathcal{C}|$ locations with one container in each location.

3. The number of lid overstows caused by the blocker transport b is
$$\#LO_{\alpha^{\mathsf{free}}}(b) = 2|\mathcal{C}||\mathcal{S}| + x_{\alpha^{\mathsf{free}}},$$
where
$$x_{\alpha^{\mathsf{free}}} = \big|\{j > |\mathcal{C}| \mid \alpha^{\mathsf{free}}(b)_{l_j^{\mathsf{o}}} \neq 0\}\big|.$$
Therefore any assignment causes at least $2|\mathcal{C}||\mathcal{S}| + x_{\alpha^{\mathsf{free}}}$ lid overstows.

4. If α^{free} places the element transport e_i in an over deck location l_j^{o}, then the number of lid overstows caused by this element transport is
$$\#LO_{\alpha^{\mathsf{free}}}(e_i) = \begin{cases} 0 & \text{if } 1 \leq j \leq |\mathcal{C}| \text{ and } s_i \in \mathcal{C}_j \\ 2 & \text{otherwise} \end{cases}.$$

Due to 3.) and 4.), the locations l_j^{o} for $j \leq |\mathcal{C}|$ are referred to as "‘cheap"’ locations whereas the location l_j^{o} for $j > |\mathcal{C}|$ are referred to as "‘expensive"’ locations. It is therefore seen that $x_{\alpha^{\mathsf{free}}}$ is the number of expensive locations that α places parts of the blocker transport in.

The above statements follows as mentioned easily from the definition of a corresponding assignment and the definition of a legal assignment. Understanding how the corresponding MLO is defined, especially the $\mathcal{O}_j$'s (see Figure 6) should give an intuition of why they are true. The statements are nevertheless proved below.

Proof:
1.): The statement follows from the fact that the set of preplaced containers has been constructed to fill up the capacity of the under deck locations:

Take an arbitrary $i \in \{1, \dots, |\mathcal{S}|\}$ and let $l_j^{\mathsf{u}} \in \mathcal{L}^{\mathsf{u}}$ be an arbitrairy under deck location. Let $s' = \max\{p \in \mathcal{O}_j \mid p \leq 3i\}$. s' exists since $1 \in \{p \in \mathcal{O}_j \mid p \leq 3i\}$. Since $p < P$ there exists an e', such that $(s', e') \in \mathcal{T}^{\mathsf{pre}}$ and $(s', e') \in \mathcal{T}_{3i}^{\mathsf{on}}$.

Using equation (7) with $l = l_j^{\mathsf{u}}$ and $p = 3i$ we get that

$$\begin{aligned} 1 = M^{l_j^{\mathsf{u}}} &\geq \sum_{t \in \mathcal{T}_{3i}^{\mathsf{on}}} \big(\alpha^{\mathsf{free}}(t)_{l_j^{\mathsf{u}}} + \alpha^{\mathsf{pre}}(t)_{l_j^{\mathsf{u}}}\big) \\ &\geq \alpha^{\mathsf{pre}}(m', n')_{l_j^{\mathsf{u}}} + \alpha^{\mathsf{free}}(e_i)_{l_j^{\mathsf{u}}} + \alpha^{\mathsf{free}}(b)_{l_j^{\mathsf{u}}} \\ &= 1 + \alpha^{\mathsf{free}}(e_i)_{l_j^{\mathsf{u}}} + \alpha^{\mathsf{free}}(b)_{l_j^{\mathsf{u}}}, \end{aligned}$$

i.e. $\alpha^{\mathsf{free}}(e_i)_{l_j^{\mathsf{u}}} + \alpha^{\mathsf{free}}(b)_{l_j^{\mathsf{u}}} = 0$. Hence the free transports $b, e_1, \dots, e_{|\mathcal{S}|}$ are not placed in an under deck location.

Next we will show that if $\alpha^{\mathsf{free}}(b)_l \neq 0$ then $\alpha^{\mathsf{free}}(e_i)_l = 0$ for all i. Let therefore $l \in \mathcal{L}$ be given and take an $i \in \{1, \dots, |\mathcal{S}|\}$. Assume that $\alpha^{\mathsf{free}}(b)_l \neq 0$. Using equation (7) with $p = 3i$ we get that

$$1 = M^l \geq \sum_{t \in \mathcal{T}_{3i}^{\mathsf{on}}} \big(\alpha^{\mathsf{free}}(t)_l + \sum_{l \in \mathcal{L}} \alpha^{\mathsf{pre}}(t)_l\big) \geq \alpha^{\mathsf{free}}(e_i) + \alpha^{\mathsf{free}}(b) \geq 1 + \alpha^{\mathsf{free}}(e_i)_l,$$

i.e. $\alpha^{\mathsf{free}}(e_i)_l = 0$.

2.): Let $e_i = (3i-2, 3i+1)$ be an element transport. Per definition $ld_{3i-2,3i+1} = 1$, so since α^{free} is a legal assignment,

$$\sum_{l\in\mathcal{L}} \alpha^{\mathsf{free}}(e_i)_l \overset{(6)}{=} 1.$$

Since all addends in the sum are non-negative, we must have that there is only one positive addend and that it equals 1, i.e. there exists exactly one $l \in \mathcal{L}$ such that $\alpha^{\mathsf{free}}(e_i)_l = 1$ while the rest is zero.

Since we have that

$$1 = M^l \geq \sum_{t\in\mathcal{T}_2^{\mathsf{on}}} \Big(\alpha^{\mathsf{free}}(t)_l + \sum_{l\in\mathcal{L}} \alpha^{\mathsf{pre}}(t)_l\Big) \geq \alpha^{\mathsf{free}}(b)_l,$$

$\alpha^{\mathsf{free}}(b)_l \leq 1$ for all l. Since

$$\sum_{l\in\mathcal{L}} \alpha^{\mathsf{free}}(b)_l \overset{(6)}{=} ld_{1P} = |\mathcal{C}|,$$

we therefore must have that b is placed in exacly $|\mathcal{C}|$ locations with one container in each location.

3.): Assume that α^{free} places b in $l_j^\circ \in \mathcal{L}^\circ$, i.e. $\alpha^{\mathsf{free}}(b)_{l_j^\circ} = 1$. Assume furthermore that l_j° is an expensive location, i.e. $j > |\mathcal{C}|$. Then

$$\mathcal{O}_j \cap]\,1, P\,[= \Big\{\{3i-1, 3i\} \mid i \in \{1,\ldots,|\mathcal{S}|\}\Big\} \cup \{P-1\},$$

and $\big|\mathcal{O}_j \cap]\,1, P\,[\,\big| = 2|\mathcal{S}| + 1$.

If $j \leq |\mathcal{C}|$ we likewise get:

$$\begin{aligned}\mathcal{O}_j \cap]\,1, P[\; &= \Big\{\{3i-1, 3i\} \;\big|\; i \in \{1,\ldots,|\mathcal{S}|\} \text{ and } s_i \notin \mathcal{C}_j\Big\}\\ &\quad \cup \Big\{\{3i-1+3|\mathcal{S}|, 3i+3|\mathcal{S}|\} \;\big|\; i \in \{1,\ldots,|\mathcal{S}|\} \text{ and } s_i \in \mathcal{C}_j\Big\}.\end{aligned}$$

The size of $\mathcal{O}_j \cap]\,1, P\,[$ is therefore

$$\big|\mathcal{O}_j \cap]\,1, P\,[\big| = 2|\mathcal{S} \setminus \mathcal{C}_j| + 2|\mathcal{C}_j| = 2|\mathcal{S}|.$$

Since the blocker transport according to 2). will be placed in exactly $|\mathcal{C}|$ locations out of which $x_{\alpha^{\mathsf{free}}}$ are expensive locations, we have that

$$\begin{aligned}\#LO_{\alpha^{\mathsf{free}}}(b) &= \sum_{j=1}^{L} \big|\mathcal{O}_j \cap]\,1, P\,[\,\big| \cdot \alpha^{\mathsf{free}}(b)_{l_j^\circ}\\ &= x_{\alpha^{\mathsf{free}}}(2|\mathcal{S}|+1) + (|\mathcal{C}| - x_{\alpha^{\mathsf{free}}}) \cdot 2|\mathcal{S}|\\ &= 2|\mathcal{C}||\mathcal{S}| + x_{\alpha^{\mathsf{free}}}.\end{aligned}$$

4.): Let $i \in \{1,\ldots,|\mathcal{S}|\}$ be given. Then $\mathcal{O}_j \cap]\,3i-2, 3i+1\,[\subseteq \{3i-1, 3i\}$.

Assume first that e_i is placed by α^{free} in $l_j^\circ \in \mathcal{L}^\circ$ with $j \leq |\mathcal{C}|$ (e_i is placed in exactly one location according to 2.) and this is an over deck location according to 1).).
According to the definition of $\mathcal{O}_j$ we have that if $s_i \in \mathcal{C}_j$ then

$$\mathcal{O}_j \cap]\,3i-2, 3i+1\,[= \emptyset$$

i.e. e_i causes zero lid overstows.

If $j \leq |\mathcal{C}|$ and $s_i \notin \mathcal{C}_j$ we get that

$$\mathcal{O}_j \cap]\, 3i-2, 3i+1\, [= \{3i-1, 3i\},$$

i.e. e_i causes 2 lid overstows. Likewise if s_i is placed in in l_j° with $j > |\mathcal{C}|$ we have that

$$\mathcal{O}_j \cap]\, 3i-2, 3i+1\, [= \{3i-1, 3i\},$$

i.e. e_i causes 2 lid overstows if it is placed in an expensive location.

Hence e_i causes 2 lid overstows if e_i is not placed in a location l_j° with $j \leq |\mathcal{C}|$ and $s_i \in \mathcal{C}_j$. □

We can now show the correspondence between the two decision problems, i.e. that an MSC* instance is a "‘yes"’-instance if and only if the corresponding MLO is a "‘yes"’-instance. This is done in the following two propositions.

Proposition 17 Let $A = \langle\, \mathcal{S},\, \mathcal{C},\, k\,\rangle$ be an MSC* instance and let B be the corresponding MLO instance. Assume there exists a cover $\mathcal{K}$ for $\mathcal{S}$ with $|\mathcal{K}| \leq k$. Then there exists a legal assignment α^{free} for B with $\#LO_{\alpha^{\mathsf{free}}} \leq k'$.

Proof: We will show that the assignment $\alpha_{\mathcal{K}}^{\mathsf{free}}$ corresponding to the provided cover has the required property.

Let $e_i \in \mathcal{T}$ be an arbitrary element transport. According to the definition of $\alpha_{\mathcal{K}}^{\mathsf{free}}$ (Definition 13), e_i is placed in a location l_j° where $j \leq |\mathcal{C}|$, $s_i \in \mathcal{C}_j$ and $\mathcal{C}_j \in \mathcal{K}$. According to Proposition 16(4), e_i does not contribute with any lid overstows. Since i is arbitrairy, this is true for all local transports.

We have by definition of $\alpha_{\mathcal{K}}^{\mathsf{free}}$ that the number of expensive locations in which the blocker transport is placed is $x_{\alpha^{\mathsf{free}}} = |\mathcal{K}|$. According to Proposition 16(3), the lid overstows caused by the blocker transport is

$$\#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}}(b) = 2|\mathcal{C}||\mathcal{S}| + x_{\alpha^{\mathsf{free}}} = 2|\mathcal{C}||\mathcal{S}| + |\mathcal{K}|.$$

Since $\alpha^{\mathsf{pre}}(t)_{l_j^{\circ}} = 0$ for all $t \in \mathcal{T}^{\mathsf{pre}}$ we have that $\#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}}(t) = 0$ for all $t \in \mathcal{T}^{\mathsf{pre}}$. The total number of lid overstows is therefore:

$$\begin{aligned}
\#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}} &= \sum_{t \in \mathcal{T}} \#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}}(t) \\
&= \#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}}(b) + \sum_{i=1}^{|S|} \#LO_{\alpha_{\mathcal{K}}^{\mathsf{free}}}(e_i) \\
&= 2|\mathcal{C}||\mathcal{S}| + |\mathcal{K}| + 0 \leq 2|\mathcal{C}||\mathcal{S}| + k \overset{\text{def. of } k'}{=} k',
\end{aligned}$$

which is what we wanted to show. □

Proposition 18 Let $A = \langle\, \mathcal{S},\, \mathcal{C},\, k\,\rangle$ be an MSC* instance and let $B = \langle\, P,\, L, LD,\, \alpha^{\mathsf{pre}},\, \mathbf{M},\, k',\rangle$ be the corresponding MLO instance.

Assume there exists a legal assignment α^{free} for B with $\#LO_{\alpha^{\mathsf{free}}} \leq k'$. Then there exists a cover $\mathcal{K}$ for A with $|\mathcal{K}| \leq k$.

Proof: Let μ be the smallest integer such that there exists a legal assignment β for B with $\#LO_{\beta} = 2|\mathcal{C}||\mathcal{S}| + \mu$. Since α^{free} exists and has $\#LO_{\alpha^{\mathsf{free}}} \leq k' = 2|\mathcal{C}||\mathcal{S}| + k$, we must have $2|\mathcal{C}||\mathcal{S}| + \mu \leq 2|\mathcal{C}||\mathcal{S}| + k$, i.e. $\mu \leq k$, and according to Proposition 16(3) μ must be larger or equal to x_{β}, i.e $x_{\beta} \leq \mu$.

To a start we will show that any element transport e_i is placed in a location l_j° where $j \leq |\mathcal{C}|$ and the corresponding element $s_i \in \mathcal{C}_j$.

Assume therefore on the contrary that there exists an element transport $e_{i'}$ that is placed by β in a location l_j° where $j > |\mathcal{C}|$ or $j \leq |\mathcal{C}|$ and $s_{i'} \notin \mathcal{C}_j$. According to 16(4) we must have that $\#LO_{\beta}(e_{i'}) = 2$. We want to reach a contradiction. Since there *does* exist a cover for $\mathcal{S}$ (A is an MSC* instance) there must be a $j' \leq |\mathcal{C}|$ such that $s_{i'} \in \mathcal{C}_{j'}$.

We must have that β places a part of the blocker transport in the corresponding over deck location, $l^{\circ}_{j'}$, because otherwise $e_{i'}$ could be moved to the location $l^{\circ}_{j'}$ where it would cause no lid overstows. I.e. we can consider the following assignment $\beta' : \mathcal{P} \times \mathcal{P} \to \mathbb{N}_0^{2L}$ defined on $\mathcal{T}^{\mathsf{free}}$ as follows:

$$\forall t \neq e_{i'} \, . \, \beta'(t) = \beta(t)$$
$$\beta'(e_{i'}) = (0, \ldots, 0, \underset{\substack{\uparrow \\ j'}}{1}, 0, \ldots, 0)$$

It is easy to see, that β' is a legal assignment and that $\#LO_{\beta'} = \#LO_{\beta} - 2$. I.e. β' causes fewer lid overstows than β, which contradicts the choice of β as the assignment causing the fewest lid overstows.

I.e. β places a part of the blocker transport in $l^{\circ}_{j'}$. However, we can still find another assignment that causes fewer lid overstows than β: We will move a part of the blocker transport from the location $l^{\circ}_{j'}$ to a vacant expensive location. Hereby we only get one extra lid overstow due to the blocker transport while we can save two lid overstow due to $e_{i'}$, since we can move this to the cheap location where part of the blocker transport was moved from.

That is, we consider the assignment β' given on $\mathcal{T}^{\mathsf{free}}$ by the following:

$$\begin{aligned} \forall t \notin \{e_{i'}, b\} \, . \, \beta'(t) = \beta(t), & \\ \beta'(e_{i'})_l = \begin{cases} 1 & \text{if } l = l^{\circ}_{j'} \\ 0 & \text{otherwise} \end{cases}, & \\ \beta'(b)_l = \begin{cases} 1 & \text{if } \beta(b)_l = 1 \text{ and } l \neq l^{\circ}_{j'} \\ 1 & \text{if } l = l^{\circ}_{\max\{r \mid \beta(b)_{l^{\circ}_r} = 0\}} \\ 0 & \text{otherwise} \end{cases}. & \end{aligned}$$

β' places local transports different from $e_{i'}$ in the same locations as β, while $e_{i'}$ is placed in $l^{\circ}_{j'}$. The blocker transport is placed where β places it, except that the part of the blocker transport that is placed by β in the mentioned $l^{\circ}_{j'}$ is moved to an expensive location (the last vacant one).

According to 16(3) and (4) we have that

$$\begin{aligned} \#LO_{\beta'}(b) &= \#LO_{\beta}(b) + 1, \\ \#LO_{\beta'}(e_{i'}) &= \#LO_{\beta}(e_{i'}) - 2, \\ \#LO_{\beta'}(e_i) &= \#LO_{\beta}(e_i) \text{ for all } i \neq i'. \end{aligned}$$

Therefore we have that β' causes fewer lid overstows:

$$\begin{aligned} \#LO_{\beta'} &= \#LO_{\beta'}(b) + \sum_{i=1}^{|\mathcal{S}|} \#LO_{\beta'}(e_i) \\ &= \#LO_{\beta}(b) + 1 + \sum_{i=1}^{|\mathcal{S}|} \#LO_{\beta}(e_i) - 2 = \#LO_{\beta} - 1, \end{aligned}$$

which again contradicts the choice of β as the assignment causing the least number of lid overstows.

Ergo we must have that for all i, e_i is placed by β in a location l°_j where $j \leq |\mathcal{C}|$ and the corresponding element $s_i \in \mathcal{C}_j$. I.e.

$$\mathcal{K} = \{\mathcal{C}_j \in \mathcal{C} \mid \exists i \, . \, \beta(e_i)_{l^{\circ}_j} = 1\}$$

is a cover of $\mathcal{S}$. A seen, $\mathcal{K}$ is the set of subsets of $\mathcal{C}$ corresponding to the cheap locations where at least one element transport is placed by β.

According to 16(2), b is placed in $|\mathcal{C}|$ locations out of which x_{β} are expensive locations, which means that b is placed in $|\mathcal{C}| - x_{\beta}$ cheap location. Since no e_i can be placed in a location where part of b is placed (16(1)), the element transports are placed in at most $|\mathcal{C}| - (|\mathcal{C}| - x_{\beta}) = x_{\beta}$ locations. Hence

$$|\mathcal{K}| \leq x_{\beta} \leq \mu \leq k,$$

i.e. $\mathcal{K}$ is the needed cover of $\mathcal{S}$ of size at most k. □

Now we are finally ready to prove the *NP*-completeness of MLO:

Proposition 19 Let Φ : MSC* $\rightarrow$ MLO be the function that maps an MSC* instance to its corresponding MLO instance.

We then have that $A \in$ MSC* is a "'yes"'-instance' if and only if $\Phi(A) \in MLO$ is a "'yes"'-instance.

Proof: Follows from Proposition 17 and Proposition 18. ☐

Theorem 20 MLO is *NP*-complete.

Proof: Since the length of the parameters of an MLO instance is a polynomial in the length of the parameters of an MSC* instance, the corresponding MLO instance can be calculated in polynomial time. To show that MLO is *NP*-complete we therefore only need to argue, that MLO is in *NP*, then the result follows from Proposition 19.

However, MLO $\in$ *NP* is true since the number of lid overstows for the MLO instance can be calculated in polynomial time for a given assignment: By definition we have that

$$\#LO_{\alpha^{\mathsf{free}}} = \sum_{j=1}^{L} \sum_{(s,e)\in\mathcal{T}} \alpha(s,e)_{l_j^{\mathsf{o}}} \cdot \left|\mathcal{O}_j \cap \,]\, s, e\,[\,\right|.$$

Both sums are over finite sets whose sizes are polynomial in the size of the MSC* instance. $\alpha^{\mathsf{free}}(s,e)_{l_j^{\mathsf{o}}}$ and $\alpha^{\mathsf{pre}}(s,e)_{l_j^{\mathsf{o}}}$ can be accessed in polynomial time if α^{pre} and α^{free} are implemented e.g. as arrays.

$\mathcal{O}_j$ can be found in polynomial time since a port p can be added to the set if

$$\sum_{q=1}^{P} \left(\alpha(q,p)_{l_j^{\mathsf{u}}} + \alpha(p,q)_{l_j^{\mathsf{u}}}\right) \neq 0.$$

Again the sum is over $\mathcal{P}$ whose size is polynomial in the length of the MSC* instance, and the addends of the sum can be accessed in polynomial time. The elements of $\mathcal{O}_j$ that are in the open interval from s to e can further be found in linear time. Thus the number of lid overstows can be calculated in polynomial time and MLO is *NP*-complete. ☐

5 Conclusion

The currently most successful approaches to container vessel stowage decompose the problem hierarchically and first generates a master plan that distribute containers to abstract storage areas of the vessel called locations. Overstowage between containers within locations can normally be ignored, but overstowage of containers between locations separated by a hatch-lid must be taken into account. In this report, we have shown that it is an *NP*-complete problem to generate master plans which minimize the number of these lid overstows.

Since master plans most likely will be central to any successful approach to container vessel stowage, this result show that future research should focus on finding efficient heuristics for generating master plans or decompose the problem further.

References

[1] D. Ambrosino, A. Sciomachen, and E. Tanfani. Stowing a containership: the master bay plan problem. *Transportation Research*, 38:81–99, 2004.

[2] D. Ambrosino, A. Sciomachen, and E. Tanfani. A decomposition heuristic for the container ship stowage problem. *Journal of Heuristics*, 12:211–133, 2006.

[3] M. Avriel, M. Penn, and N. Shpirer. Container ship stowage problem: Complexity and connection to the coloring of circle graphs. *Discrete Applied Mathematics*, 103:271–279, 2000.

[4] M. Avriel, M. Penn, N. Shpirer, and S. Witteboon. Stowage planning for container ships to reduce the number of shifts. *Annals of Operations Research*, 76:55–71, 1998.

[5] R. C. Botter and M. A. Brinati. Stowage container planning: A model for getting an optimal solution. In *Proceedings of the Seventh International Conference on Computer Applications in the Automation of Shipyard Operation and Ship Design*, pages 217–229. North-Holland Publishing Co., 1992.

[6] O. Dubrovsky, G. Levitin, and M. Penn. A genetic algorithm with a compact solution encoding for the container ship stowage problem. *Journal of Heuristics*, 8:585–599, 2002.

[7] Michael R. Garey and David S. Johnson. *Computers and Intractability - A Guide to the Theory of NP-Completeness*. W.H.Freeman and Company, 1979.

[8] P. Giemesch and A. Jellinghaus. Optimization models for the containership stowage problem. In *Proceedings of the International Conference of the German Operations Research Society*, 2003.

[9] J. G. Kang and Y. D. Kim. Stowage planning in maritime container transportation. *Journal of the Operational Research Society*, 53:415–426, 2002.

[10] M. Levinson. *The Box: How the Shipping Container Made the World Smaller and the World Economy Bigger*. Princeton University Press, 2008.

[11] I. D. Wilson and P. A. Roach. Principles of combinatorial optimization applied to container-ship stowage planning. *Journal of Heuristics*, 5:403–418, 1999.