On the maximum number of edges in chordal graphs of bounded degree and matching number

Jean R. S. Blair · **Pinar Heggernes** · **Paloma T. Lima** · **Daniel Lokshtanov**

Abstract We determine the maximum number of edges that a chordal graph G can have if its degree, $\Delta(G)$, and its matching number, $\nu(G)$, are bounded. To do so, we show that for every $d, \nu \in \mathbb{N}$, there exists a chordal graph G with $\Delta(G) < d$ and $\nu(G) < \nu$ whose number of edges matches the upper bound, while having a simple structure: G is a disjoint union of cliques and stars.

Keywords chordal graphs; maximum number of edges; matching number.

1 Introduction

A problem that dates back to 1960 is to determine the maximum number of edges that a graph can have if its maximum degree and matching number are each bounded. It is important to note that this problem does not impose any constraint on the number of vertices of the graph. Because of that, in general, if one of the two parameters is not bounded, there is no upper bound on the number of edges that a graph can have. One can simply construct graphs formed by stars (trees that have only a single vertex of degree greater than one) or single edges. A star with unbounded number of leaves has matching number one but unbounded degree, while a graph that is a disjoint union of an unbounded number of edges has bounded degree but unbounded matching number. By Vizing's Theorem, every graph can have its edge set partitioned into a family of at most $\Delta(G)+1$ matchings, where $\Delta(G)$ denotes the degree of the graph G. Thus, bounding both the maximum degree and the matching number is actually enough to bound the number of edges that a graph can have. Chvátal and Hanson [7] gave a tight upper bound on this value, in the case where no further restrictions are imposed to the graphs considered. Later on, Balachandran and Khare [1] gave a constructive proof

J. R. S. Blair
Department of Electrical Engineering and Computer Science, United States Military Academy, West Point, New York, USA
E-mail: jean.blair@westpoint.edu

P. Heggernes
University of Bergen, Bergen, Norway
E-mail: pinar.heggernes@uib.no

P. T. Lima
IT University of Copenhagen, Copenhagen, Denmark
E-mail: palt@itu.dk

D. Lokshtanov
Department of Computer Science, University of California Santa Barbara, USA
E-mail: daniello@ucsb.edu

of the same result, which made it possible to identify the structure of the graphs achieving the given bound on the number of edges. Such graphs are called edge-extremal graphs. In some cases, they contain induced subgraphs isomorphic to stars, as well as to cycles of length four.

An interesting problem that arises from these results is to investigate how the number of edges in the edge-extremal graphs is affected if we impose some additional structural property on the graphs considered. More specifically, what happens if we restrict the question to graph classes in which cycles of length four or stars are forbidden induced subgraphs? Natural candidates for such graph classes are chordal graphs, that is, graphs without induced cycles of length at least four, and claw-free graphs. In the past few years, bounds for this problem have indeed been established for claw-free graphs in the work of Dibek et al. [8]. Furthermore, the problem has been resolved on other graph classes, such as bipartite graphs, split graphs, disjoint unions of split graphs and unit interval graphs in the work of Måland [14]. However, on chordal graphs, the problem had so far remained unresolved. Chordal graphs form an extremely well-studied graph class, both from a structural and from an algorithmic point of view, with many and various applications.

In this work, we determine the maximum number of edges that a chordal graph can have, given the constraints on its maximum degree and matching number. Given $d,\nu \in \mathbb{N}$, we denote by $\mathscr{M}_{chordal}(d,\nu)$ the set of chordal graphs such that $\Delta(G) < d$ and $\nu(G) < \nu$. A graph in $\mathscr{M}_{chordal}(d,\nu)$ achieving this maximum number of edges is called an edge-extremal graph. In order to establish the upper bound on the number of edges of an edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu)$ we show that, among them, there is one that has a very simple structure: it is a disjoint union of cliques and stars of a given size.

Theorem 1 *There exists an edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars.*

Section 3 is entirely devoted to the proof of Theorem 1. Once the structure of this special edge-extremal graph is known, we are able to establish the following upper bound on the number of edges of a graph in $\mathscr{M}_{chordal}(d,\nu)$.

Theorem 2 *Given $d,\nu \in \mathbb{N}$, the maximum number of edges of a graph in $\mathscr{M}_{chordal}(d,\nu)$ is given by:*

$$\begin{cases} (d-1)(\nu-1), & \text{if } d \text{ is even} \\ (d-1)(\nu-1) + \lfloor \frac{d-1}{2} \rfloor \lfloor \frac{\nu-1}{\lceil \frac{d-1}{2} \rceil} \rfloor, & \text{if } d \text{ is odd} \end{cases}$$

Moreover, a graph achieving this number of edges is

$$\begin{cases} (\nu-1)K_{1,d-1}, & \text{if } d \text{ is even} \\ rK_{1,d-1} + qK_d, & \text{if } d \text{ is odd,} \end{cases}$$

where $\nu - 1 = q\lceil \frac{d-1}{2} \rceil + r$, with $r \geq 0$.

We also show that this result is tight in the sense that the same bound does not hold for any superclass of chordal graphs that is defined by a finite collection of forbidden induced cycles. It is worth mentioning that this problem is related to the famous problem of computing Ramsey numbers, the general case being equivalent to determining Ramsey numbers for line graphs [2]. A preliminary version of this work appeared in the proceedings of LATIN 2020 [3].

2 Preliminaries

The graphs considered are simple and undirected. We denote by V_G and E_G the vertex set and edge set of G, respectively. Given $x \in V_G$, we denote by $N_G(x)$ the neighborhood of x, that is, the set of vertices that are adjacent to x. Two vertices $x, y \in V_G$ are *true twins* if $N_G(x) \cup \{x\} = N_G(y) \cup \{y\}$. Given $x \in V_G$ and $X \subseteq V_G \setminus \{x\}$, we say x is *universal* to X if $X \subseteq N_G(x)$. For a set $X \subset V_G$, $N_G(X)$ denotes the set of vertices in $V_G \setminus X$ that have at least one neighbor in X. The *degree* of x is denoted by $\deg_G(x)$ and is defined as $|N_G(x)|$. The *degree of a graph* G is the maximum degree of a vertex in G and it is denoted by $\Delta(G)$. A vertex x is a *leaf* of G if $\deg_G(x) = 1$.

Given $S \subseteq V_G$, the *subgraph induced by* S is denoted by $G[S]$, and has S as its vertex set and $\{uv \mid u, v \in S \text{ and } uv \in E_G\}$ as its edge set. A *clique* is a set $K \subseteq V_G$ such that $G[K]$ is a complete graph. A clique is *maximal* if it is not properly contained in another clique. An *independent set* is a set S such that $G[S]$ has no edges. A vertex $v \in V_G$ is a *simplicial vertex* if $N_G(v)$ is a clique. Given a set $S \subseteq V_G$, we denote the graph $G[V_G \setminus S]$ by $G \setminus S$. If $S = \{v\}$, we denote the graph $G[V_G \setminus \{v\}]$ simply by $G \setminus v$. The set S is a *separator* if $G \setminus S$ has a larger number of connected components than G. Given a set $F \subseteq E_G$, the *subgraph induced by* F is denoted by $G[F]$, and has the endpoints of the edges in F as its vertex set and F as its edge set.

A set $M \subseteq E_G$ is a *matching* if no two edges in M share a common vertex and M is a *perfect matching* if every vertex of V_G is the endpoint of an edge in M. The *matching number of* G, denoted by $\nu(G)$, is the largest size of a matching in G. A graph G is a *factor-critical graph* if for every $v \in V_G$, $G \setminus v$ has a perfect matching.

Given a family $\mathscr{H}$ of graphs, we say that G is an *$\mathscr{H}$-free graph* if G does not contain an induced subgraph that is isomorphic to a graph in $\mathscr{H}$. If $\mathscr{H} = \{H\}$, we say G is an H-free graph. A *tree* is a connected acyclic graph. A *star* is a tree with at most one vertex that is not a leaf, and for $k \in \mathbb{N}$, a *k-star*, denoted by $K_{1,k}$, is a star with k leaves. A graph is a *complete graph* on n vertices, denoted by K_n, if there is an edge between every pair of its vertices. Given two graphs G and H, the *disjoint union of G and H*, denoted by $G + H$ is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. We denote by rH the graph that is the disjoint union of r copies of a graph H. A graph G is a *bipartite graph* if V_G can be partitioned into two independent sets. A bipartite graph with bipartition (A, B) is a *chain graph* if there exists an ordering $v_1 v_2 \dots v_r$ of the vertices of A such that $N_G(v_r) \subseteq \dots \subseteq N_G(v_1)$. This property of the vertices of A is called the *nested neighborhood* property. Bipartite chain graphs are also known to be the bipartite $2K_2$-free graphs.

A graph is a *chordal graph* if it has no induced cycle of length at least four. Chordal graphs constitute a widely studied graph class, with many different characterisations. Given a graph G, a *clique tree* of G is a tree $\mathscr{T}$ such that every vertex of $\mathscr{T}$ is a maximal clique of G and for every $v \in V(G)$, $T_v = \{A \in V_{\mathscr{T}} \mid v \in A\}$ induces a subtree of $\mathscr{T}$. The vertices of $\mathscr{T}$ are referred to as *bags* and denoted with capital letters. For simplicity, we denote the set of vertices of G associated with a vertex of $\mathscr{T}$ with the same capital letter. A characterisation of chordal graphs due to Gavril [11] states that a graph is chordal if and only if it has a clique tree. One important property of clique trees is that, if $\mathscr{T}$ is a clique tree of a chordal graph G and $AB \in E_{\mathscr{T}}$, then $A \cap B$ is a separator for the graph G. Another important characterisation of chordal graphs is concerned with vertex orderings and simplicial vertices. An ordering $v_1 v_2 \dots v_n$ of the vertices of G is a *perfect elimination ordering* for G if for every i, the vertex v_i is simplicial in the graph $G[\{v_{i+1}, \dots, v_n\}]$. A characterisation of chordal graphs due to Fulkerson and Gross [10] states that a graph is chordal if and only if it has a perfect elimination ordering. See [4] for an overview of the properties of chordal graphs and clique trees.

Given two integers d and ν and a graph class $\mathscr{C}$, we denote by $\mathscr{M}_{\mathscr{C}}(d, \nu)$ the set of all graphs G in $\mathscr{C}$ such that $\Delta(G) < d$ and $\nu(G) < \nu$. A graph in $\mathscr{M}_{\mathscr{C}}(d, \nu)$ that has the maximum number of edges is called an *edge-extremal graph*. When the graph class considered is the class of all graphs, we write simply $\mathscr{M}(d,\nu)$. The following lemma establishes a connection between edge-extremal graphs and factor-critical graphs in some graph classes. Even though the statement we present here is different from the one stated in [1], the proof in [1] suffices to prove the result as stated below.

Lemma 1 ([1]) *Let $\mathscr{C}$ be a graph class that is closed under vertex deletion and closed under taking disjoint union with stars. Let G be an edge-extremal graph in $\mathscr{M}_{\mathscr{C}}(d,\nu)$ with maximum number of connected components that are $(d-1)$-stars. Then every connected component of G that is not a $(d-1)$-star is factor-critical.*

The following statement gives a summary of the results obtained by Balachandran and Khare [1].

Theorem 3 ([1]) *Given $d,\nu \in \mathbb{N}$, the maximum number of edges of a graph in $\mathscr{M}(d,\nu)$ is given by $(d-1)(\nu-1)+\lfloor\frac{d-1}{2}\rfloor\lfloor\frac{\nu-1}{\lceil\frac{d-1}{2}\rceil}\rceil$. Moreover, a graph achieving this number of edges is*

$$\begin{cases} rK_{1,d-1}+qK'_d, & \text{if } d \text{ is even} \\ rK_{1,d-1}+qK_d, & \text{if } d \text{ is odd,} \end{cases}$$

where $\nu-1=q\lceil\frac{d-1}{2}\rceil+r$, with $r\geq 0$, and K'_d is the graph obtained from K_d by the removal of the edges of a perfect matching and addition of a new vertex adjacent to $d-1$ vertices.

In Section 3, we show the corresponding bounds for $\mathscr{M}_{chordal}(d,\nu)$ and obtain graphs that achieve these bounds. We remark that, in Theorem 3, the graph $rK_{1,d-1}+qK_d$, obtained when d is odd, is already a chordal graph. Thus, for odd d, the edge-extremal chordal graphs have the same number of edges as the edge-extremal general graphs. Our proof, however, does not rely on this fact and has a unified approach, that works regardless of the parity of d.

3 Chordal graphs

In this section we present our main result. The strategy to determine the maximum number of edges that a graph in $\mathscr{M}_{chordal}(d,\nu)$ can have is to show that among the edge-extremal graphs in $\mathscr{M}_{chordal}(d,\nu)$, there is one that has a very simple structure: it is a disjoint union of cliques and stars of a given size.

Theorem 1 (restated) *There exists an edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars.*

Overview of the proof. The proof is by contradiction. We start with an edge-extremal graph of $\mathscr{M}_{chordal}(d,\nu)$ that is, in some sense, closest to being a disjoint union of cliques and stars. From that, we will perform a series of modifications in the graph in order to obtain another graph of $\mathscr{M}_{chordal}(d,\nu)$ that has at least as many edges as the one we started with, but that is closer to being a disjoint union of cliques and stars, which will be a contradiction with our initial choice. To perform the modifications, we will consider a specific clique tree of our edge-extremal graph and exploit the structure of this graph around one of its cliques, given by a carefully chosen node of the tree. A crucial part of the proof is to ensure that, after each modification, the obtained graph still belongs to $\mathscr{M}_{chordal}(d,\nu)$. In this vein, Lemmas 3 and 4 will precisely show that the two modifications we describe can indeed be performed without disrupting membership in $\mathscr{M}_{chordal}(d,\nu)$. In this way, we obtain a new edge-extremal graph that, as a result, has several structural properties that will be exploited to conclude the proof.

Proof of Theorem 1 Assume for a contradiction that there is no edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu)$ that is a disjoint union of cliques and stars. Let W be an edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu)$ with maximum number of $(d-1)$-stars and subject to that, with maximum number of connected components. Let W' be a connected component of W that is not a clique nor a star and let $\nu_1 = \nu(W')+1$. By Lemma 1, W' is a factor-critical graph and therefore $|V_{W'}| = 2\nu_1 - 1$. Note that $W' \in \mathscr{M}_{chordal}(d,\nu_1)$ and, in fact, W' is edge-extremal in $\mathscr{M}_{chordal}(d,\nu_1)$. Indeed, if this was not the case, we would be able to obtain a graph in $\mathscr{M}_{chordal}(d,\nu)$ with more edges than W by replacing the connected component W' by an edge-extremal graph of $\mathscr{M}_{chordal}(d,\nu_1)$. Among all the edge-extremal graphs in $\mathscr{M}_{chordal}(d,\nu_1)$ with $2\nu_1 - 1$ vertices, let G be the one that has a clique tree with minimum number of leaves. Note that, in particular, G is connected, by the maximality of the number of connected components of the graph W.

Let $\mathscr{T}$ be a clique tree of G achieving the minimum number of leaves. We consider $\mathscr{T}$ rooted in an arbitrary bag R. Let X be a node of $\mathscr{T}$. We denote by T_X the subtree of $\mathscr{T}$ rooted at the node X. We define a subgraph G_X associated with each node X of $\mathscr{T}$ in the following way. If $X = R$, then $G_X = G$. Otherwise, let S be the separator of G given by the intersection between X and its parent in $\mathscr{T}$ and let V_{T_X} be the set of vertices appearing in the bags of T_X. The subgraph G_X associated with the node X is given by $G[V_{T_X} \setminus S]$. Observe that if X is a leaf of $\mathscr{T}$, then G_X is a complete graph. Let B be a bottommost bag in $\mathscr{T}$ such that G_B is not a complete graph. Note that such a node indeed exists since G is not a complete graph itself. Let $B_1, \ldots, B_k$ be the children of B in $\mathscr{T}$ and let $S_i = B \cap B_i$. Note that, by our choice of B, the graph $G_{B_i} = G[V_{T_{B_i}} \setminus S_i]$ is a complete graph for every i. For simplicity, from now on we denote $C_i = V_{T_{B_i}} \setminus S_i$ and hence $G_{B_i} = G[C_i]$.

We start with the following two observations stating how the vertices of C_i are connected to those of B and what is the structure of the tree T_{B_i}.

Observation 1 For every i, the subgraph of G induced by the edges $E_i = \{xy \mid x \in S_i \text{ and } y \in C_i\}$ is a chain graph.

Proof. Note that (S_i, C_i) constitutes a partition of $V_{T_{B_i}}$, thus $G[E_i]$ is a bipartite graph. Suppose for a contradiction that there exists an induced $2K_2$ in $G[E_i]$ with vertex set $\{x_1, y_1, x_2, y_2\}$, with $x_1, x_2 \in S_i$ and $y_1, y_2 \in C_i$. Since S_i and C_i are cliques in G, the vertices x_1, y_1, x_2 and y_2 would form an induced C_4 in G, a contradiction with the fact that G is chordal. Therefore $G[E_i]$ is indeed bipartite and $2K_2$-free, that is, a chain graph. ⌟

Observation 2 For every i, the subtree T_{B_i} is a path.

Proof. Since $G[V_{T_{B_i}}]$ is a chordal graph, by Observation 1, the bipartite graph obtained from $G[V_{T_{B_i}}]$ by deleting the edges inside S_i and C_i is a chain graph. Because of the nested neighborhood property of chain graphs, $G[V_{T_{B_i}}]$ has a clique tree that is a path. Since $\mathscr{T}$ was chosen with minimum number of leaves, the subtree T_{B_i} is a path, for every i. ⌟

In what follows, we want to modify the graph G in such a way to obtain a graph that is still chordal, has the same number of vertices as G and belongs to $\mathscr{M}_{chordal}(d,\nu_1)$, but either has more edges than G, or is disconnected, or has a clique tree with fewer leaves. Either one of these outcomes will contradict the choice of G. The modifications to be performed in G will consist in the addition and removal of edges, as well as of vertices. After each modification, one crucial part of the proof is to ensure that the matching number of the obtained graph is still strictly less than ν_1. This will follow from the fact that G has $2\nu_1 - 1$ vertices. Therefore, the addition of edges to G does not lead to a graph with matching number greater or equal to ν_1. Moreover, for any $k \in \mathbb{N}$, the same holds for the simultaneous removal of k vertices from such a graph followed by simultaneous addition of k new vertices. We formalize this in the following observation for later reference.

Observation 3 Let H be any graph on $2\nu_1 - 1$ vertices. Then any modification that preserves the number of vertices of H cannot lead to a graph with matching number at least ν_1.

For every $v \in B$, let $f_G(v,i)$ denote the number of neighbors that vertex v has in the clique C_i, that is, $f_G(v,i) = |N_G(v) \cap C_i|$. Note that if $f_G(v,i) > 0$, then $v \in S_i$. Let $u_{i,1}, \dots, u_{i,|C_i|}$ be an ordering of the vertices of C_i such that $\deg_G(u_{i,1}) \geq \deg_G(u_{i,2}) \geq \dots \geq \deg_G(u_{i,|C_i|})$. Since $G[E_i]$ is a chain graph by Observation 1, we may assume that for every $v \in B$ with $f_G(v,i) > 0$, $N_G(v) \cap C_i = \{u_{i,1}, \dots, u_{i,f_G(v,i)}\}$.

We first state and prove the following lemma that can be understood as the converse of Observation 1 and that will be useful throughout the paper to show that a graph is chordal.

Lemma 2 *Let H be any graph and $B, C_1, \dots, C_k$ be cliques of H such that*

- *$N_H(C_i) \subseteq B$, for every $1 \leq i \leq k$;*
- *$H[V_H \setminus (\cup_{i=1}^{k} C_i)]$ is a chordal graph.*

If the subgraph G_i of H induced by the edges $E_i = \{xy \mid x \in B \text{ and } y \in C_i\}$ is a chain graph for every $1 \leq i \leq k$, then H is a chordal graph.

Proof. Since G_i is a chain graph and $\deg_G(u_{i,1}) \geq \deg_G(u_{i,2}) \geq \dots \geq \deg_G(u_{i,|C_i|})$, we conclude that $N_{G_i}(u_{i,|C_i|}) \subseteq N_{G_i}(u_{i,|C_i|-1}) \subseteq \dots \subseteq N_{G_i}(u_{i,1})$. We will show how to construct a perfect elimination ordering for the graph H. Note that for every $1 \leq i \leq k$, the vertex $u_{i,|C_i|}$ is simplicial in H. Moreover, for every $i \leq k$ and every $j \leq |C_i|$, the vertex $u_{i,j}$ is simplicial in $H[V_H \setminus \{u_{i,j+1}, \dots, u_{i,|C_i|}\}]$. Indeed, since E_i is a chain graph, the set $\{u_{i,1}, \dots, u_{i,j}\} \cup (B \cap N_H(u_{i,j}))$ is a clique. Finally, since $H[V_H \setminus (\cup_{i=1}^{k} C_i)]$ is a chordal graph, it has a perfect elimination ordering σ' of its vertices. Let $\sigma_i = u_{i,|C_i|} \dots u_{i,1}$. Then $\sigma_1 \sigma_2 \dots \sigma_k \sigma'$ is a perfect elimination ordering for H, which concludes the proof that H is chordal. ⌟

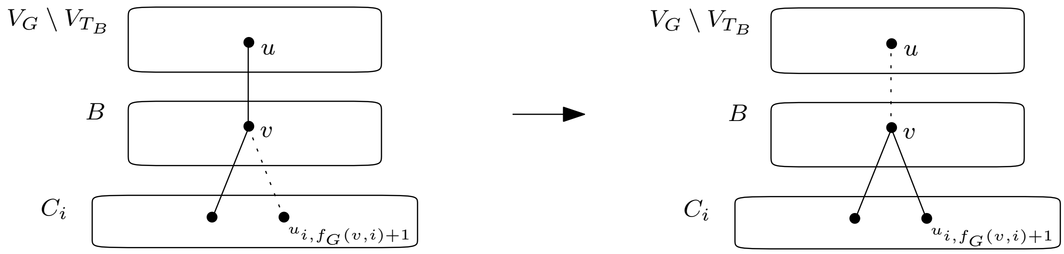
We are now ready to state the two modifications that will be used repeatedly throughout the proof of Theorem 1.

Modification 1 Let $B, C_1, \dots, C_k$ be subsets of the vertex set of the chordal graph G as previously described and let $v \in B$. For $1 \leq i \leq k$, if $0 < f_G(v,i) < |C_i|$ and v has a neighbor that does not belong to $G[V_{T_B}]$, we do the following (see Figure 1a):

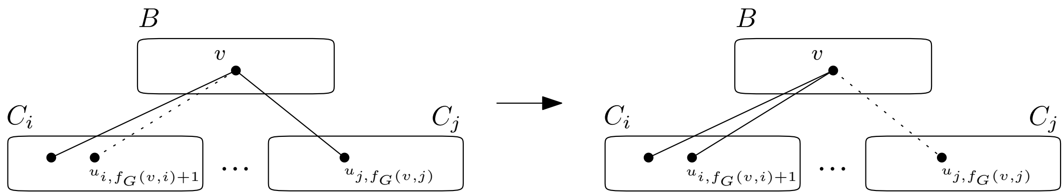
(i) Add an edge between v and the vertex $u_{i,f_G(v,i)+1}$;
(ii) Delete the edge from v to one of its neighbors outside $G[V_{T_B}]$. This neighbor is chosen in the following way: consider the subtree T_v of $\mathscr{T}$ formed by the bags that contain the vertex v. Let L be a leaf of T_v that is not in the subtree rooted in B. Such a leaf exists since v has a neighbor outside $G[V_{T_B}]$. Let L' be the bag that is adjacent to L in T_v. Since $L \nsubseteq L'$, there exists $u \in L \setminus L'$. Let u be the chosen neighbor of v and delete the edge uv.

Lemma 3 *Modification 1 preserves both membership in $\mathscr{M}_{chordal}(d, \nu_1)$ and number of edges.*

Proof. Let G' be the graph obtained with the application of Modification 1. First, note that since the edges between B and C_i induce a chain graph, $v \in N_G(u_{i,f_G(v,i)})$ and $v \notin N_G(u_{i,f_G(v,i)+1})$, we have that $N_G(u_{i,f_G(v,i)+1}) \subset N_G(u_{i,f_G(v,i)})$. Therefore, in G' we have $N_{G'}(u_{i,f_G(v,i)+1}) \subseteq N_{G'}(u_{i,f_G(v,i)})$, which implies that the addition of the edge $vu_{i,f_G(v,i)+1}$ preserves the nested neighborhood property in the bipartite graph induced by the edges between B and C_i. Thus, by Lemma 2 and since G is chordal, the addition of this edge does not disrupt membership in the class of chordal graphs. Therefore, to show that G' is chordal it suffices to show that the removal of the edge uv preserves chordality. We do so by providing a clique tree to $G - uv$. This clique tree is obtained from $\mathscr{T}$ as follows. Let $L'' = L \setminus \{u\}$. If $L'' \neq L'$, add L'' between L and L' in the tree $\mathscr{T}$ and delete v from L. If $L'' = L'$, just delete v from L in $\mathscr{T}$. Also, note that this operation does not change the number of leaves in $\mathscr{T}$. Hence, we obtain that the graph G' is chordal. Note that the degree of v does not change



(a) Modification 1



(b) Modification 2

Fig. 1: The dotted lines between two vertices indicate non-edges.

with this modification. The only vertex whose degree was increased by Modification 1 is $u_{i,f_G(v,i)+1}$. However, note that since $N_{G'}(u_{i,f_G(v,i)+1}) \subset N_{G'}(u_{i,f_G(v,i)})$ and $\deg_{G'}(u_{i,f_G(v,i)}) = \deg_G(u_{i,f_G(v,i)}) < d$, we have that $\deg_{G'}(u_{i,f_G(v,i)+1}) < d$. This shows that Modification 1 does not increase the maximum degree of the graph. Since Modification 1 preserves the number of vertices and $|V_G| = 2\nu_1 - 1$, by Observation 3, it does not lead to a graph with matching number greater or equal to ν_1. We conclude the proof by observing that $|E_{G'}| = |E_G|$, since exactly one edge was deleted and exactly one edge was added by this modification. ⌟

Modification 2 Let $B, C_1, \ldots, C_k$ be subsets of the vertex set of the chordal graph G as previously described and let $v \in B$. For $1 \leq i \leq k$, if $0 < f_G(v,i) < |C_i|$ and $f_G(v,j) > 0$ for some $j > i$, we do the following (see Figure 1b):

(i) Delete the edge $vu_{j,f_G(v,j)}$;
(ii) Add the edge $vu_{i,f_G(v,i)+1}$.

Lemma 4 *Modification 2 preserves both membership in $\mathscr{M}_{chordal}(d,\nu_1)$ and number of edges.*

Proof. Let G' be the graph obtained after the application of Modification 2. The only vertex that had its degree increased by this modification is $u_{i,f_G(v,i)+1}$. However, as in the proof of Lemma 3, since $vu_{i,f_G(v,i)} \in E_G$ and $vu_{i,f_G(v,i)+1} \notin E_G$, we have that $\deg_G(u_{i,f_G(v,i)+1}) < \deg_G(u_{i,f_G(v,i)})$. Thus, $\deg_{G'}(u_{i,f_G(v,i)+1}) \leq \deg_{G'}(u_{i,f_G(v,i)}) = \deg_G(u_{i,f_G(v,i)}) < d$, implying that $\Delta(G') < d$. Moreover, again since Modification 2 preserves the number of vertices and $|V_G| = 2\nu_1 - 1$, by Observation 3, it does not lead to a graph with matching number greater or equal to ν_1. It is also easy to see that $|E_{G'}| = |E_G|$, since exactly one edge was deleted in step (i) and exactly one edge was added to the graph in step (ii). It remains to show that the obtained graph is still chordal. Indeed, note that the deletion of the edge $vu_{j,f_G(v,j)}$ (resp. addition of the edge $vu_{i,f_G(v,i)+1}$) preserves the nested neighborhood property in the bipartite graph induced by the edges between B and C_j (resp. C_i). Thus, by Lemma 2, the graph G' is a chordal graph. ⌟

Recall that our graph G is an edge-extremal graph in $\mathscr{M}_{chordal}(d,\nu_1)$, and that the graph obtained from W by replacing the connected component W' by G is also an edge-extremal graph in

$\mathscr{M}_{chordal}(d,v)$ with maximum number of connected components. For simplicity of notation, we call this graph W again. Let G^* be the graph obtained from G by exhaustive applications of Modification 2 followed by exhaustive applications of Modification 1. It follows immediately from Lemmas 3 and 4 that $G^* \in \mathscr{M}_{chordal}(d,v_1)$ and that G^* is edge-extremal in this set. Moreover, if the graph obtained after the application of any modification is disconnected, we reach a contradiction with the maximality of the number of components of W. Therefore, we can assume G^* is connected. The following lemma describes the major structural property of G^* that will be exploited in the remainder of the proof.

Lemma 5 *Let G^* be the graph obtained from G by exhaustive applications of Modification 2 followed by exhaustive applications of Modification 1. Then, for every $v \in V_{G^*} \cap B$ and every i, if v has at least one neighbor in C_i, one of the following conditions hold:*

(a) $C_i \subseteq N_{G^*}(v)$;
(b) $\deg_{G^*}(v) = \Delta(G^*)$ *and* $N_{G^*}(v) \subseteq B \cup C_1 \cup \ldots \cup C_i$.

Proof. First, let G' be the graph obtained from G by exhaustive applications of Modification 2. Since this modification can no longer be applied, then for every $v \in B$ and every i such that $f_{G'}(v,i) > 0$, we have that either $f_{G'}(v,i) = |C_i|$ or $f_{G'}(v,j) = 0$ for every $j > i$. Thus, for every $v \in B$, there exists at most one index ℓ such that $0 < f_{G'}(v,\ell) < |C_\ell|$. Now we apply Modification 1 exhaustively to G' and obtain the graph G^*. Recall that $f_{G'}(v,i) = |N_{G'}(v) \cap C_i|$. Observe that, for every $v \in B$, if $f_{G'}(v,i) = 0$, then $f_{G^*}(v,i) = 0$ and if $f_{G'}(v,i) = |C_i|$, then $f_{G^*}(v,i) = |C_i|$. Indeed, Modification 1 is only applied to a vertex $v \in B$ and index i if $0 < f_{G'}(v,i) < |C_i|$ and, when applied, it does not change $f_{G'}(v,j)$ for $j \neq i$. Furthermore, since Modification 1 can no longer be applied, if a vertex v is such that $0 < f_{G^*}(v,i) < |C_i|$, then v has no neighbors outside $B \cup C_1 \cup \ldots \cup C_i$. That is, if condition (a) does not hold, then $N_{G^*}(v) \subseteq B \cup C_1 \cup \ldots \cup C_i$. It remains to show that, in this case, $\deg_{G^*}(v) = \Delta(G^*)$. To see this, first note that by Lemmas 3 and 4, we have that $|E_{G^*}| = |E_G|$ and that $G^* \in \mathscr{M}_{chordal}(d,v_1)$, thus G^* is an edge-extremal graph in $\mathscr{M}_{chordal}(d,v_1)$. If $\deg_{G^*}(v) < \Delta(G^*)$, we can add to G^* the edge $vu_{i,f_{G^*}(v,i)+1}$. The addition of this edge does not change the maximum degree of G^* since $\deg_{G^*}(v) < \Delta(G^*)$ by assumption, and $d_{G^*}(u_{i,f_{G^*}(v,i)+1}) \leq d_{G^*}(u_{i,f_{G^*}(v,i)}) < \Delta(G^*)$. Moreover, by Lemma 2, the addition of this edge preserves chordality, and together with Observation 3, we conclude that it preserves membership in $\mathscr{M}_{chordal}(d,v_1)$. However, the obtained graph has more edges than G^*, a contradiction. This shows that $\deg_{G^*}(v) = \Delta(G^*)$ and concludes the proof of Lemma 5. ⌟

Since the graph G^* is such that $\Delta(G^*) < d$ and $|E_{G^*}| = |E_G|$, we can replace the connected component G in our edge-extremal graph W by G^*. This replacement will be convenient since Lemma 5 provides useful information on the structure of G^*. More concretely, *in the rest of the proof we shall assume that $B, C_1, \ldots, C_k$ satisfy the conclusion of Lemma 5.*

Let b be the size of the clique B, let $\Delta = \Delta(G^*)$ and recall that S_i is the separator between the bag B_i and B and S is the separator between the bag B and its parent in $\mathscr{T}$. We are now going to conclude the proof of Theorem 1 with a case analysis.

Case 1: There exists i such that $|C_i| + b \leq \Delta + 1$.

Case 1.1: $k \geq 2$.

We may assume, without loss of generality, that $|C_1| \leq |C_2| \leq \ldots \leq |C_k|$. In particular, this implies that $|C_1| + b \leq \Delta + 1$. We will show that, in this case, all the vertices of C_1 are adjacent to all the vertices of $S_1 \cup \ldots \cup S_k$. This will lead to a contradiction with the number of leaves of the clique tree of G. Suppose for a contradiction that there exists $v \in S_1 \cup \ldots \cup S_k$ that is not universal to C_1. This implies that $f_{G^*}(v,1) < |C_1|$.

We will show that the graph G^* can be modified in order to obtain another edge-extremal graph, also in $\mathscr{M}_{chordal}(d,v_1)$, in which v is adjacent to every vertex of C_1.

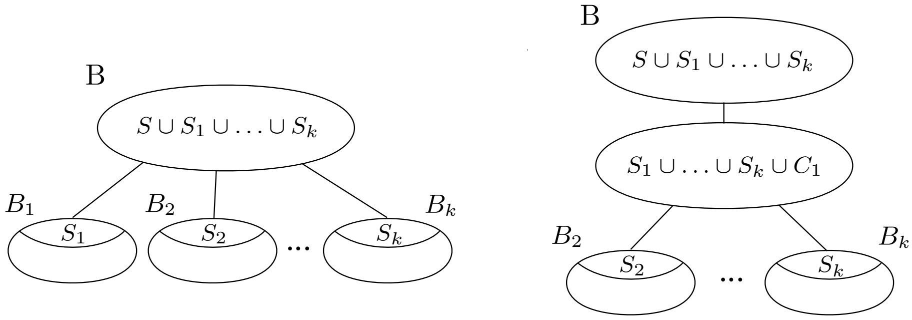


Fig. 2: To the left, the clique tree $\mathscr{T}$ and to the right, a clique tree of the updated graph G^* that has fewer leaves than $\mathscr{T}$.

First, note that it cannot be the case that $f_{G^*}(v,1) > 0$, since by Lemma 5, if $0 < f_{G^*}(v,1) < |C_1|$, then v has maximum degree and has no neighbors outside $B \cup C_1$. However, this is a contradiction, since $|C_1| + b \leq \Delta + 1$. Thus, we conclude that $f_{G^*}(v,1) = 0$.

In what follows, we will modify the graph G^* and the deletion of some edges might disrupt the membership in the class of chordal graphs. In these cases, we will use the following modification in order to restore it.

Modification 3 Let H be any graph satisfying the conditions of Lemma 2. We do the following:

(i) Delete from H all the edges xy such that $x \in B$ and $y \in C_i$ for some i;
(ii) For each $v \in B$ and each $1 \leq i \leq k$, if $f_H(v,i) > 0$, add the edges between v and the vertices $u_{i,1}, \ldots, u_{i,f_H(v,i)}$.

Lemma 6 *Modification 3 preserves membership in the class of chordal graphs and number of edges.*

Proof. Let H' be the graph obtained from H by Modification 3. We show that, for every $1 \leq i \leq k$, $H'[E_i]$ is a chain graph, where $E_i = \{xy \mid x \in B \text{ and } y \in C_i\}$. It will then follow from Lemma 2 that H' is chordal. Suppose this is not the case, and let $v,w \in B$ and $u_{i,j}, u_{i,\ell} \in C_i$, with $j < \ell$, be such that $\{v,w,u_{i,j},u_{i,\ell}\}$ induces a $2K_2$ in $H'[E_i]$ with edges $vu_{i,j}$ and $wu_{i,\ell}$. However, since $j < \ell$, the edge $wu_{i,j}$ was also added in step (ii), a contradiction. Finally, it is easy to see that $|E_{H'}| = |E_H|$, since the degrees of the vertices in B remain unchanged. ⌟

We now modify G^* as follows. Let j be the largest index for which $f_{G^*}(v,j) > 0$. If $f_{G^*}(v,j) = |C_j|$, since $|C_1| \leq |C_j|$, we can delete $|C_1|$ edges between v and C_j and add all the edges between v and C_1. We then apply Modification 3 to the obtained graph in order to obtain a graph that, by Lemma 6, is chordal. Note that the only vertices whose degree has increased are the ones in C_1. However, since $|C_1| + b \leq \Delta + 1$, we conclude that the maximum degree of G^* did not increase.

If $f_{G^*}(v,j) < |C_j|$, then, by Lemma 5, v has maximum degree and has no neighbors outside $B \cup C_1 \cup \ldots \cup C_j$. Furthermore, recall that $f_{G^*}(v,1) = 0$, which means that v has no neighbors in C_1. Since $|B| = b$ and $v \in B$, we have that v has exactly $\Delta - b + 1$ neighbors in $C_2 \cup \ldots \cup C_j$. Therefore $\sum_{\ell=2}^{j} f_{G^*}(v,\ell) = \Delta - b + 1$. Since $|C_1| \leq \Delta - b + 1$ by assumption, we can delete $|C_1|$ edges between v and vertices of $C_2 \cup \ldots \cup C_j$ and add all the edges between v and C_1. We then apply Modification 3 to the obtained graph in order to obtain a graph that, by Lemma 6, is chordal. Again, the only vertices whose degree has increased in this process are the ones from C_1, thus we conclude the obtained graph still has degree at most Δ.

Finally note that in both cases, the modifications do not change the number of edges of G^*, since $\sum_{\ell=1}^{k} f_{G^*}(v,\ell)$ remains the same. They also preserve the number of vertices of G^*, which has $2\nu_1 - 1$

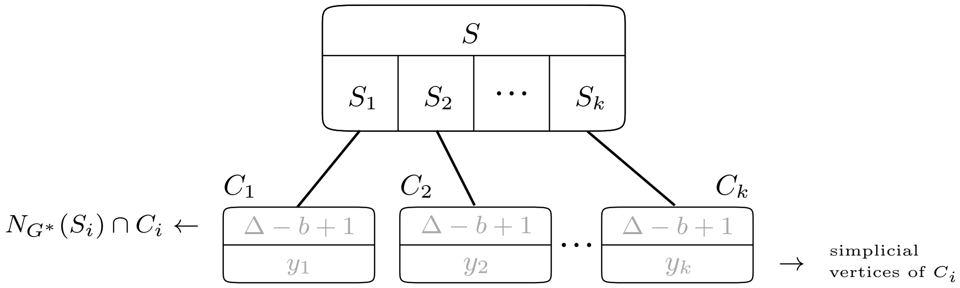


Fig. 3: Graph G^* in case 2. Thick lines indicate all possible edges between the sets. Gray text indicates the cardinality of the vertex set. Note that every vertex of C_i that does not belong to $N_{G^*}(S_i)$ is a simplicial vertex of G^*.

vertices. Thus by Observation 3, the modifications lead to a graph with matching number still strictly smaller than ν_1. Moreover, in this obtained graph, v is adjacent to all the vertices of C_1. We perform this change for every $v \in S_1 \cup \ldots \cup S_k$ such that $f_{G^*}(v, 1) > 0$ and obtain a new edge-extremal graph in $\mathscr{M}_{chordal}(d, \nu_1)$ such that all the vertices of C_1 are adjacent to all the vertices of $S_1 \cup \ldots \cup S_k$. Recall that among all the edge-extremal graphs in $\mathscr{M}_{chordal}(d, \nu_1)$ with $2\nu_1 - 1$ vertices, G was the one that had a clique tree with minimum number of leaves. This new graph, however, has a clique tree that has fewer leaves than the clique tree $\mathscr{T}$ of G. This is because the clique $C_1 \cup S_1 \cup \ldots \cup S_k$ is contained in B and contains the intersection between B and each child of B (see Figure 2). This contradicts the minimality of the number of leaves of $\mathscr{T}$.

Case 1.2: $k = 1$.

Since G_B is not a clique by assumption, there exists $v \in V_{G_B} \setminus S$ that is not universal to C_1 in G^*. If $f_{G^*}(v, 1) > 0$ then $v \in S_1$. In this case, by Lemma 5, v has maximum degree and no neighbors outside $B \cup C_1$. Hence, $\deg_{G^*}(v) \leq b - 1 + |C_1| - 1$, which implies that $\Delta \leq b + |C_1| - 2$. This is a contradiction with the assumption of Case 1 that $|C_1| + b \leq \Delta + 1$.

If $f_{G^*}(v, 1) = 0$, then v is a simplicial vertex in G^*. Since $v \in B$, $\deg_{G^*}(v) = b - 1$. By the assumption of Case 1, $b - 1 + |C_1| \leq \Delta$. Thus, we can add all the edges between v and the vertices of C_1 and obtain a graph whose maximum degree is still at most Δ. Moreover, by Observation 3 the matching number of the obtained graph is still less than ν_1. Finally, since v is now adjacent to all vertices of C_1, the bipartite graph induced by the edges between B and C_1 is a chain graph, and hence, by Lemma 2, the obtained graph is chordal. Thus, this graph belongs to $\mathscr{M}_{chordal}(d, \nu_1)$ and has more edges than G^*, a contradiction.

Case 2: For every i, $|C_i| + b > \Delta + 1$.

Let $v \in S_1 \cup \ldots \cup S_k$. Let a_v be the smallest index such that $f_{G^*}(v, a_v) > 0$. Note that v cannot be universal to C_{a_v} in G^*, since by assumption $|C_{a_v}| + b > \Delta + 1$. By Lemma 5, $\deg_{G^*}(v) = \Delta$ and $N_{G^*}(v) \subseteq B \cup C_{a_v}$. This implies that for every $v \in S_1 \cup \ldots \cup S_k$, there exists a unique index a_v such that $f_{G^*}(v, a_v) > 0$. That is, for any $j \neq a_v$, $f_{G^*}(v, j) = 0$, and thus $S_i \cap S_j = \emptyset$ if $i \neq j$. Also, since $N_{G^*}(v) \subseteq B \cup C_{a_v}$ and v has degree Δ, we have that $f_{G^*}(v, a_v) = \Delta - b + 1$. That is, if $a_v = a_u$, then u and v are true twins in G^*. Moreover, for any $1 \leq i < j \leq k$, $|N_{G^*}(S_i) \cap C_i| = |N_{G^*}(S_j) \cap C_j|$. Let S be the separator between B and its parent in the clique tree $\mathscr{T}$. Since for every $v \in S_1 \cup \ldots \cup S_k$, $N_{G^*}(v) \subseteq B \cup C_{a_v}$, we know that $S \cap S_i = \emptyset$, for every i. Also, since the graph G^* is connected, $S \neq \emptyset$. See Figure 3.

Let $u \in N_{G^*}(S_i) \cap C_i$. Suppose for a contradiction that $\deg_{G^*}(u) < \Delta$. Let G_1 be the graph obtained from G^* by the deletion of one vertex of S and addition of a new vertex w in S_i, such that $N_{G_1}[w] = B \cup (N(S_i) \cap C_i)$.

Claim 1 *G_1 is chordal, $|E_{G_1}| \geq |E_{G^*}|$ and $\Delta(G_1) = \Delta(G^*)$.*

Proof. Note that w is a true twin of the vertices in S_i, and since the vertices of S_i have maximum degree, it holds that $\deg_{G_1}(w) = \Delta$ and hence the graph G_1 has at least as many edges as G^*. The only vertices whose degree has increased after this modification are those belonging to $N_{G_1}(S_i) \cap C_i$. However, note that if $x \in N_{G_1}(S_i) \cap C_i$, then $\deg_{G_1}(x) = \deg_{G^*}(x) + 1$. Since $\deg_{G^*}(x) < \Delta$ by assumption, we have that $\deg_{G_1}(x) \leq \Delta(G^*)$ and thus $\Delta(G_1) = \Delta(G)$. To see that G_1 is chordal, it suffices to notice that the class of chordal graphs is closed under vertex deletion and under the addition of true twins. ⌟

If G_1 is disconnected or has more edges than G^*, we have a contradiction. We repeat the above modification until either the graph obtained is disconnected, that is, until $S = \emptyset$, or until for every i, the degree of the vertices in $N_{G_1}(S_i) \cap C_i$ is Δ. Let G_2 be the graph obtained after exhaustive application of the above modification. If G_2 is disconnected, we have a contradiction with the maximality of the number of connected components of our initial edge-extremal graph. Otherwise, by Claim 1, we have that G_2 is chordal, $|E_{G_2}| \geq |E_{G^*}|$ and $\Delta(G_2) = \Delta(G^*)$. Moreover, by Observation 3, $\nu(G_2) < \nu_1$. Therefore, we can now replace G^* by G_2 in our edge-extremal graph W. Note that G_2 is such that:

1. For every $1 \leq i < j \leq k$, $S_i \cap S_j = \emptyset$;
2. For every $1 \leq i \leq k$, the vertices of S_i and of $N_{G_2}(S_i) \cap C_i$ have degree Δ and $|N_{G_2}(S_i) \cap C_i| = \Delta - b + 1$.

Case 2.1: $k \geq 2$.

Let y_i be the number of simplicial vertices in the clique C_i. Assume without loss of generality that $y_1 \geq y_2$. We perform the following modifications in the graph G_2: deletion of one simplicial vertex from C_2 and one vertex from S_1 and addition of one vertex to S_2 and one simplicial vertex to C_1. Note that, after this modification, the only vertices that had their degree changed are the simplicial vertices from C_1 and C_2. Since these simplicial vertices did not have maximum degree before, the degree of the obtained graph does not exceed the degree of G_2. Note that $y_2 - 1 + \Delta - b + 1 + \Delta$ edges were removed by the deletion of the two vertices and $y_1 + \Delta - b + 1 + \Delta$ were added by the addition of the other two vertices. However, since $y_1 \geq y_2$, we have that the obtained graph has strictly more edges than G_2, which is a contradiction.

Case 2.2: $k = 1$.

Since all vertices in S_1 and in $N_{G_2}(S_1) \cap C_1$ have maximum degree, we can perform the following modification in G_2: delete all vertices of S_1 and add $|S_1|$ vertices to $N_{G_2}(S_1) \cap C_1$. The graph obtained after this modification has the same number of edges as G_2, since $|S_1|$ vertices of degree Δ were removed and the same amount of vertices with the same degree was added. However, the obtained graph is disconnected, which is a contradiction with the maximality of the number of connected components of the edge-extremal graph W.

This concludes the proof of Theorem 1. □

By Theorem 1, we know that there is an edge-extremal graph in $\mathscr{M}_{chordal}(d, \nu)$ that is a disjoint union of cliques and stars. The next lemma gives a tight upper bound on the number of edges of such an edge-extremal graph when d is even.

Lemma 7 *Let G be a graph in $\mathscr{M}_{chordal}(d, \nu)$ that is a disjoint union of cliques and stars. If d is even, then $|E_G| \leq (d-1)(\nu - 1)$.*

Proof. Let G be a graph such that $\Delta(G) \leq d-1$ and $\nu(G) \leq \nu - 1$ and that is a disjoint union of cliques and stars. We proceed by induction in k, the number of connected components of G. If $k = 1$ and G is a star, then $\Delta(G) \leq d-1$ and $|E_G| \leq d-1$. If G is a clique, then $|V_G| \leq d$. If $|V_G| = d$, then $\Delta(G) = d-1$ and $\nu(G) = \frac{d}{2}$, since G is even. Then $|E_G| = \binom{|V_G|}{2} = \frac{d(d-1)}{2} = \Delta(G)\nu(G) \leq (d-1)(\nu-1)$. Now assume $|V_G| \leq d-1$. Since G is a clique, $\nu(G) \geq \frac{|V_G|-1}{2}$. Hence, $|E_G| = \frac{|V_G|(|V_G|-1)}{2} \leq (d-1)\nu(G) \leq (d-1)(\nu-1)$.

Let G be a disjoint union of cliques and stars with $k > 1$ connected components. Let H be a component of G and G' be the graph obtained from G by the removal of the vertices of H. Then $\Delta(G') \leq \Delta(G)$ and by the induction hypothesis, $|E_{G'}| \leq \Delta(G')\nu(G')$. If H is a star, then $\nu(G') = \nu(G) - 1$ and $|E_G| \leq |E_{G'}| + \Delta(G)$, which implies that $|E_G| \leq \Delta(G)\nu(G) \leq (d-1)(\nu-1)$.

If H is a clique, we have that $|E_G| = |E_{G'}| + \binom{|V_H|}{2} \leq \Delta(G')\nu(G') + \binom{|V_H|}{2}$. If $|V_H| = d$, then $\nu(H) = \frac{d}{2}$, since d is even, and $\nu(G') = \nu(G) - \frac{d}{2}$. Hence, $|E_G| \leq \Delta(G)(\nu(G) - \frac{d}{2}) + \binom{d}{2} \leq (d-1)(\nu-1)$. Now assume that $|V_H| \leq d-1$. Since G is a clique, $\nu(H) \geq \frac{|V_H|-1}{2}$ and thus $\nu(G') \leq \nu(G) - \frac{|V_H|-1}{2}$. Hence $|E_G| \leq \Delta(G')\nu(G') + \binom{|V_H|}{2} \leq \Delta(G)(\nu(G) - \frac{|V_H|-1}{2}) + \binom{|V_H|}{2}$. And since $|V_H| \leq d-1$, we conclude that $|E_G| \leq (d-1)(\nu-1)$. ⌟

By Theorem 3, we already know the maximum number of edges that a graph that is a disjoint union of cliques and stars can have when d is odd. From Theorem 1 and Lemma 7, we obtain our main result, Theorem 2 (see page 2), which establishes the upper bound on the number of edges that a chordal graph of $\mathscr{M}_{chordal}(d,\nu)$ can have and shows that the obtained bound is tight.

4 Final remarks and open problems

In this work, we determined the maximum number of edges that a chordal graph can have if its maximum degree and matching number are bounded. We also exhibit examples of graphs achieving this bound.

An interesting question that remains open comes from the fact that the graph K_i' used in Theorem 3 has an induced C_4. For each d and ν, what is the maximum number of edges of a graph in $\mathscr{M}_{C_4\text{-}free}(d,\nu)$? We point out that the bound on the number of edges for chordal graphs does not hold for C_4-free graphs, as can be seen by the graph P, obtained from the famous Petersen graph by the subdivision of one edge (see Figure 4). We have that $\Delta(P) = 3$, $\nu(P) = 5$ and $|E_P| = 16$. The bound given by Theorem 1 when $d = 4$ and $\nu = 6$ is 15. This idea can be further generalized to create examples in the class of $\mathscr{H}$-free graphs, where $\mathscr{H}$ is any finite collection of cycles. Indeed, let r be the size of a largest cycle of $\mathscr{H}$. A result due to Kochol [13] about snarks implies that for any $r \geq 5$ there exists an infinite family of 3-regular graphs of girth r that have a perfect matching. Let G be one such graph and let H be the graph obtained from G by the subdivision of one edge. The graph H is clearly $\mathscr{H}$-free and is such that $\Delta(H) = 3$, $\nu(H) = \nu(G)$ and $|E_H| = 3\nu(H) + 1$, while the bound given by Theorem 1 when $d = 4$ and $\nu = \nu(H) + 1$ is $3\nu(H)$.

Another related open problem mentioned in [8] is to determine the maximum number of edges that an arbitrary connected graph can have if its degree and matching number are bounded. We remark that the problem is open even for connected chordal graphs. In this case, we observe that the edge-extremal graph described in Theorem 1 can be turned into a connected graph by identifying two leaves of distinct components. This shows that the maximum number of edges does not change when the connectivity contraint is imposed to chordal graphs and d is even. However, this is not the case when d is odd. In particular, as shown in [1], if $\nu - 1$ divides $\frac{d-1}{2}$, the edge-extremal graphs described in Theorem 3 are unique and thus, the connectivity contraint will definitely result in a decrease in the maximum number of edges.

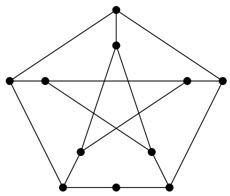


Fig. 4: A C_4-free graph with $\Delta = 3$, $\nu = 5$ and $|E| = 16$.

It is also interesting to point out that, as briefly hinted in the introduction, the problem of determining the maximum number of edges a graph can have under constraints on its degree and matching number is related to that of edge coloring graphs. An edge coloring of a graph is a partition of its edge set into disjoint matchings (also referred to as color classes). By Vizing's Theorem, a graph can always be edge colored with $\Delta(G)+1$ colors. However, the problem of deciding whether $\Delta(G)$ colors suffice is NP-complete [12]. There are very few known sufficient conditions to guarantee a graph cannot be edge colored with $\Delta(G)$ colors. The most famous (and simple) of them is to test whether the graph has *too many edges*, that is, if $|E_G| > \Delta(G)\lfloor \frac{|V_G|}{2} \rfloor$. A graph whose number of edges satisfy this inequality is called *overfull*. In particular, if a graph has a vertex v of maximum degree such that $G[N_G[v]]$ is overfull, then this graph is called *neighborhood overfull* and cannot be edge colored with $\Delta(G)$ colors either. The complexity of the edge coloring problem restricted to chordal graphs has remained open for many years, despite numerous efforts towards a solution (see, e.g., [5,6,9]). In particular, Figueiredo et al. [9] conjectured that a chordal graph is edge colorable with $\Delta(G)$ colors if and only if it is not subgraph overfull. The bound on the number of edges we provide in Theorem 1 implies that *no chordal graph of odd maximum degree is subgraph overfull*. Hence, if the conjecture of Figueiredo et al. holds, every such graph can be edge colored with $\Delta(G)$ colors. So far, this has only been confirmed for split graphs of odd maximum degree [6]. The same question for chordal graphs remains an interesting open problem to be solved.

Conflict of interest

The authors declare that they have no conflict of interest.

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